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Analytical Study of Structure of a Mamdani Fuzzy Controller with Three Input Variables

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8.1 INTRODUCTION

Fuzzy controllers have been used worldwide in many fields [6, 9, 20]. They are usually practically constructed, instead of theoretically designed, using the trial-and-error method and computer simulation. This is due to relatively weak fuzzy control theory in existence and also due to the fact that fuzzy controllers are generally non-linear controllers that are difficult to analyze and design. In recent years, efforts have been made to understand fuzzy controller structures (e.g. [3], [5], [8], [12–15], [19], [23], [26]), to analyze fuzzy controllers (e.g. [10], [11], [16–18], [22], [25], [28], [29]) and to develop methods for determining the stability of fuzzy control systems (e.g. [4], [7], [11], [17], [25]).

One of the most important aspects of fuzzy control research is the study of fuzzy controller structure. For general SISO (single-input–single-output) and MIMO (multiple-input–multiple-output) fuzzy controllers, we have found their analytical structures and limiting structures [24, 27]. The fuzzy controllers can always be decomposed into the sum of a global non-linear controller and a local non-linear controller. The structure of the global and local controllers depends on the configuration of the fuzzy controllers and hence is different from one controller to another. We have investigated some specific configurations in which the fuzzy controllers use error and rate change of error (rate, for short) as input variables. We have analytically proved that the structure of the non-linear fuzzy controllers that use triangular input fuzzy sets, singleton output fuzzy sets, linear fuzzy control rules, the Zadeh fuzzy logic AND operator, the Lukasiewicz fuzzy logic OR operator and the centroid defuzzifier is the sum of a global two-dimensional multilevel relay and a local nonlinear PI controller [23]. We have extended this result to other SISO fuzzy controllers and also to some MIMO fuzzy controllers [26], all of which use two input variables.

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The most popular controller in industry is the linear PID (proportional-integral-derivative) controller, which uses error, rate and derivative of rate change of error (*d*_rate, for short) of process output as input variables. In contrast, the majority of fuzzy control research and applications focuses only on fuzzy controllers that use error and rate as input variables; *d*_rate is hardly utilized. Consequently, there exist few results on the analytical structure of non-linear fuzzy controllers that use the same input variables as the PID controller does.

In this chapter we investigate the structure of a non-linear fuzzy controller using these three input variables. The controller employs triangular input fuzzy sets, trapezoidal output fuzzy sets, linear fuzzy control rules, the product fuzzy logic AND operator, the Lukasiewicz fuzzy logic OR operator, the Mamdani minimum inference method and the centriod defuzzifier. The chapter is organized as follows. In Section 8.2, the configuration and components of the fuzzy controller are defined. In Section 8.3, the analytical structure of the fuzzy controller is derived, and the characteristics of the structure are analyzed. For comparison, the structure of the fuzzy controller using error and rate as input variables is also revealed. The limiting structure of the fuzzy controller when the number of the input fuzzy sets approaches infinity is studied.

8.2 CONFIGURATION OF THE FUZZY CONTROLLER

The fuzzy controller in this study uses the process output, denoted $y(nT)$, where T is the sampling period and nT is the sampling time, to calculate its three scaled input variables. They are scaled error, scaled rate and scaled *d*_rate, as expressed mathematically by the following:

$$e^* = GE \cdot e(nT) = GE(SP(nT) - y(nT)),$$
$$r^* = GR \cdot r(nT) = GR(e(nT) - e(nT - T)),$$
$$d^* = GD \cdot d(nT) = GD(r(nT) - r(nT - T)),$$

where $SP(nT)$ is the target of process output, and $nT-T$ is the previous sampling time. GE, GR and GD are the scaling factors for error, rate and *d*_rate, respectively.

Each of the scaled input variables is fuzzified by $N = 2J + 1 (J \geq 1)$ input fuzzy sets. Among them, J fuzzy sets are for positive scaled input variables, another J fuzzy set for negative scaled input variables and one fuzzy set for near zero scaled input variables. We designate E_i, R_j and $D_k(-J \leq i, j, k \leq J)$ as a fuzzy set for e^*, r^* and d^*, respectively. Their membership functions are identical and triangular-shaped. Note that the identity of the membership functions is with respect to e^*, r^* and d^* (the scaled input variables), not with respect to $e(nT)$, $r(nT)$ and $d(nT)$ (the physical input variables). Thus, by choosing different values for GE, GR and GD, the membership functions are different in terms of $e(nT)$, $r(nT)$ and $d(nT)$. In other words, the identity requirements on the membership functions of the input fuzzy sets are actually not restrictive.

The definitions of these membership functions are shown graphically in Figure 8.1. The central values for E_{-J}, R_{-J} and D_{-J} are defined as $-L$ and those for E_J, R_J and D_J as L. The central values of the remaining $2J-1$ fuzzy sets are required to be equally spaced with the space between two adjacent sets being

$$S = \frac{L}{J}.$$

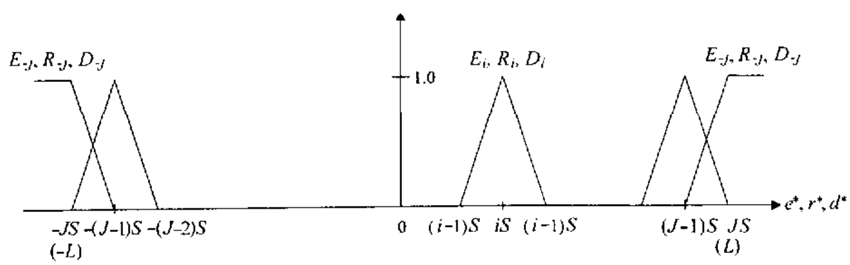

Figure 8.1 Graphical definitions of the triangular input fuzzy sets. There are a total $2J + 1$ of them

The central values of E_i, R_j and D_k are, therefore, $i \cdot S, j \cdot S$ and $k \cdot S$, respectively. That is, the membership value of the triangular fuzzy sets is one at the central values. The triangular membership functions in $[-L, L]$ are as follows: the membership function of E_i (or R_i or D_i) is zero at $(i-1)S$ and increases linearly to one at $i \cdot S$ and then decreases linearly to zero at $(i+1)S$; the membership function is zero elsewhere. In $(-\infty, -L]$ the membership functions of E_{-J}, R_{-J} and D_{-J} are always equal to one, and in $[L, +\infty)$ the membership functions of E_J, R_J and D_J are always equal to one. In these two intervals, the membership functions of all the other fuzzy sets are zero.

We designate the incremental output of the fuzzy controller at nT as $\Delta u(nT)$. There are $6J + 1$ (i.e. $3N-2$) output fuzzy sets for it, and their definitions are illustrated in Figure 8.2. Among them, $3J$ output fuzzy sets are for positive $\Delta u(nT)$, another $3J$ output fuzzy sets for negative $\Delta u(nT)$ and one output fuzzy set for near zero $\Delta u(nT)$. We designate ΔU_m, where $-3J \leq m \leq 3J$, as an output fuzzy set. The central values for ΔU_{-J} and ΔU_J are defined as $-H$ and H, respectively. The central values for the rest of the output fuzzy sets are required to be equally spaced and the space between two neighboring output fuzzy sets is

$$V = \frac{H}{3J}.$$

Obviously, the central value for ΔU_m is $m \cdot V$. The membership functions of the ΔU_m's are identical and trapezoidal-shaped with the upper-side being $2A$ and the lower-side being $2V$. To define the shape of these trapezoids, a parameter

$$\theta = \frac{A}{V}$$

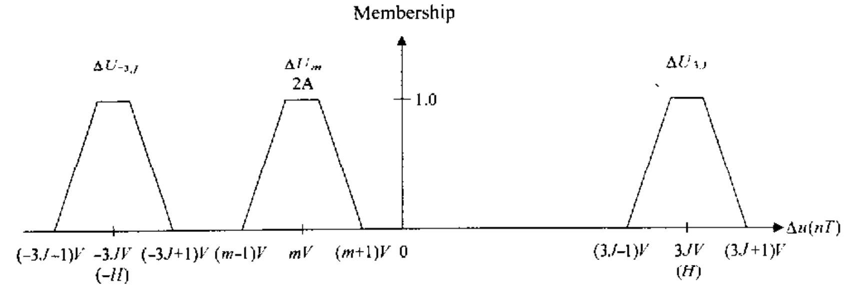

Figure 8.2 Illustrative definitions of the trapezoidal output fuzzy sets. There are a total of $6J + 1$ of them. Note that $2A$ and $2V$ are, respectively, the upper-side and lower-side of the trapezoids

is employed. We apply the following constraint to this parameter:

$$\theta \leq 0.5$$

to avoid overlap between the upper-sides of two adjacent output fuzzy sets. In $[-H, H]$ the membership function of ΔU_m increases linearly from zero at $(m-1)V$ to one at $m \cdot V - A$ and stays at one until $m \cdot V + A$ where it begins to decrease linearly to zero at $(m+1)V$. The membership function is zero elsewhere. In $(-\infty, -H]$ the membership function of ΔU_{-3J} is always equal to one, and in $[H, +\infty)$ the membership function of ΔU_{3J} is always equal to one. The membership functions of all the other output fuzzy sets are zero in these two intervals.

N^3 control rules are necessary in order to cover $N \times N \times N$ possible combinations of the input fuzzy sets. In this study we use the following linear control rules [23]:

$$\textbf{IF } e^* \text{ is } E_i \text{ AND } r^* \text{ is } R_j \text{ AND } d^* \text{ is } D_k \textbf{ Then } \Delta u(nT) \text{ is } \Delta U_{i+j+k}. \tag{8.1}$$

The control rules are said to be linear because the linear function is used to relate the indices of E, R and D to the index of ΔU.

The widely-used Mamdani minimum inference method, whose definition is illustrated in Figure 8.3, is employed to infer output fuzzy sets from input ones in the control rules. We denote by $\mu_{i,j,k}(\Delta u)$ as membership for ΔU_{i+j+k}, which is calculated by evaluating the memberships in the rule antecedents in (8.1) using the product fuzzy logic AND operator. More specifically, $\mu_{i,j,k}(\Delta u) = \mu_i(e^*)\mu_j(r^*)\mu_k(d^*)$. The shaded area in Figure 8.3 is

$$S(\mu_{i,j,k}(\Delta u)) = (2 - \mu_{i,j,k}(\Delta u) + \theta \cdot \mu_{i,j,k}(\Delta u))\mu_{i,j,k}(\Delta u) \cdot V. \tag{8.2}$$

Some fuzzy rules generate memberships for the same output fuzzy sets. In these cases, the Lukasiewicz fuzzy logic OR operator (i.e. x OR y OR z = $\min(x + y + z, 1)$) is used to obtain a combined membership because the conditions being ORed are maximally negatively correlated [21].

The prevalent centriod defuzzifier is employed to defuzzify the output fuzzy sets. Since the shapes of the output fuzzy sets are identical, the global centroid can be calculated using the local centroid, which are the products of the central values and the partial trapezoids of the

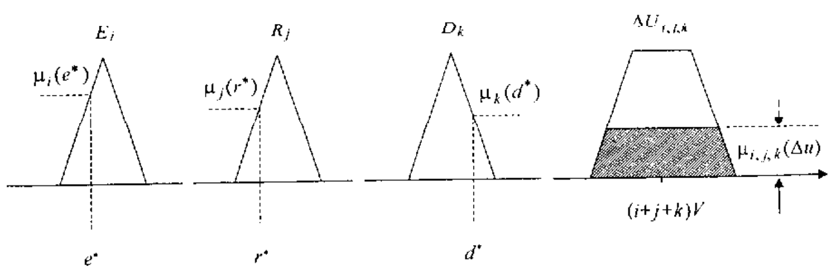

Figure 8.3 Illustrative definition of the Mamdani minimum inference method. We denote $\mu_{i,j,k}(\Delta u)$ as the membership function for the output fuzzy set ΔU_{i+j+k}. $\mu_{i,j,k}(\Delta u)$ is calculated by using the product fuzzy logic AND operator, from the memberships of E_i, R_j and D_k and the linear control rule (8.1). And $\mu_{i,j,k}(\Delta u) = \mu_i(e^*)\mu_j(r^*)\mu_k(d^*)$. The shaded area can be calculated using (8.2). Note that $2A$ and $2V$ are, respectively, the upper-side and lower-side of the trapezoidal $\Delta U_{i+j+k}(-J \leq i, j, k \leq J)$

output fuzzy sets involved. As a result, the scaled incremental output of the fuzzy controller is

$$GU \cdot \Delta u(nT) = GU \frac{\sum_{\mu_{i,j,k}(\Delta u)\neq 0} S(\mu_{i,j,k}(\Delta u)) \cdot (i+j+k)V}{\sum_{\mu_{i,j,k}(\Delta u)\neq 0} S(\mu_{i,j,k}(\Delta u))}; \tag{8.3}$$

where GU is a scaling factor for $\Delta\, u(nT)$.

8.3 ANALYTICAL STUDY OF THE FUZZY CONTROLLER STRUCTURE

Theorem 1: *The structure of the fuzzy controller using error, rate and d* rate *as input variables is the sum of a global three-dimensional multilevel relay and two local non-linear controllers, one of which is a non-linear PID controller with variable gains.*

Proof Without loss of generality, we assume

$$i \cdot S \le e^* \le (i+1)S,$$
$$j \cdot S \le r^* \le (j+1)S,$$
$$k \cdot S \le d^* \le (k+1)S,$$

where

$$-L \le e^*, r^*, d^* \le L$$

By fuzzifying e^*, r^* and d^*, the memberships for $E_i, E_{i+1}, R_j, R_{j+1}, D_k$ and D_{k+1} are

$$\mu_i(e^*) = -\frac{E^* - 0.5S}{S}, \qquad \mu_{i+1}(e^*) = \frac{E^* + 0.5S}{S}; \tag{8.4}$$

$$\mu_j(r^*) = -\frac{R^* - 0.5S}{S}, \qquad \mu_{j+1}(r^*) = \frac{R^* + 0.5S}{S}, \tag{8.5}$$

$$\mu_k(d^*) = -\frac{D^* - 0.5S}{S}, \qquad \mu_{k+1}(d^*) = \frac{D^* + 0.5S}{S} \tag{8.6}$$

where

$$E^* = e^* - (i+0.5)S, \qquad R^* = r^* - (j+0.5)S \quad \text{and} \quad D^* = d^* - (k+0.5)S. \tag{8.7}$$

Note that

$$\mu_i(e^*) + \mu_{i+1}(e^*) = 1 \qquad \mu_j(r^*) + \mu_{j+1}(r^*) = 1, \qquad \mu_k(d^*) + \mu_{k+1}(d^*) = 1.$$

The membership value for all the other input fuzzy sets is zero. Consequently, only the following eight (i.e. 2^3) fuzzy rules are executed:

If e^* is E_{i+1} AND r^* is R_{j+1} AND d^* is D_{k+1} **Then** $\Delta u(nT)$ is $\Delta U_{i+j+k+3}$. (r_1)

If e^* is E_{i+1} AND r^* is R_{j+1} AND d^* is D_k **Then** $\Delta u(nT)$ is $\Delta U_{i+j+k+2}$; (r_2)

If e^* is E_{i+1} AND r^* is R_j AND d^* is D_{k+1} **Then** $\Delta u(nT)$ is $\Delta U_{i+j+k+2}$, (r_3)

If e^* is E_{i+1} AND r^* is R_j AND d^* is D_k **Then** $\Delta u(nT)$ is $\Delta U_{i+j+k+1}$ (r_4)

If e^* is E_i AND r^* is R_{j+1} AND d^* is D_{k+1} **Then** $\Delta u(nT)$ is $\Delta U_{i+j+k+2}$ (r_5)

If e^* is E_i AND r^* is R_{j+1} AND d^* is D_k **Then** $\Delta u(nT)$ is $\Delta U_{i+j+k+1}$ (r_6)

If e^* is E_i AND r^* is R_j AND d^* is D_{k+1} **Then** $\Delta u(nT)$ is $\Delta U_{i+j+k+1}$ (r_7)

If e^* is E_i AND r^* is R_j AND d^* is D_k **Then** $\Delta u(nT)$ is ΔU_{i+j+k}. (r_8)

Using the product fuzzy logic AND operator and the membership functions (8.4–8.6), the memberships for the rule consequent in (r_1 - r_8) can be obtained as analytical expressions. Note that the rules r_2, r_3 and r_5 produce memberships for the same output fuzzy set, $\Delta U_{i+j+k+2}$, and that r_4, r_6 and r_7 generate memberships for the same output fuzzy set, $\Delta U_{i+j+k+1}$. The Lukasiewicz fuzzy logic OR operator is used to calculate the combined memberships. The combined membership for $\Delta U_{i+j+k+2}$ is

$$\min(\mu_{i+1}(e^*)\mu_{j+1}(r^*)\mu_k(d^*) + \mu_{i+1}(e^*)\mu_j(r^*)\mu_{k+1}(d^*) + \mu_i(e^*)\mu_{j+1}(r^*)\mu_{k+1}(d^*), 1). \tag{8.8}$$

Because

$$\mu_{j+1}(r^*)\mu_k(d^*) + \mu_j(r^*)\mu_{k+1}(d^*) = \frac{1}{2} - \frac{2R^*D^*}{S^2} \le 1$$

and

$$\mu_i(e^*)\mu_{j+1}(r^*)\mu_{k+1}(d^*) \le \mu_i(e^*),$$

we get

$$\begin{aligned}
&\mu_{i-1}(e^*)\mu_{j+1}(r^*)\mu_k(d^*) + \mu_{i+1}(e^*)\mu_j(r^*)\mu_{k+1}(d^*) + \mu_i(e^*)\mu_{j+1}(r^*)\mu_{k+1}(d^*) \\
&= \mu_{i+1}(e^*)(\mu_{j+1}(r^*)\mu_k(d^*) + \mu_j(r^*)\mu_{k+1}(d^*)) + \mu_i(e^*)\mu_{j+1}(r^*)\mu_{k+1}(d^*) \\
&< \mu_{i+1}(e^*) + \mu_i(e^*)\mu_{j+1}(r^*)\mu_{k+1}(d^*) \le \mu_{i+1}(e^*) + \mu_i(e^*) = 1.
\end{aligned}$$

Hence, the result of (8.8) is always the memberships being ORed. Similarly, it can be proved that this is also the case for the combined membership for $\Delta U_{i+j+k+1}$.

Substituting all the membership expressions for the output fuzzy sets into the defuzzifier (8.3) and then mathematically manipulating the resulting expressions, the analytical structure of the fuzzy controller is obtained as follows:

$$\begin{aligned}
GU \cdot \Delta u(nT) &= (i+j+k+1.5)GU \cdot V \\
&+ \left[K_i\left(e(nT) - \frac{(i+0.5)S}{GE}\right) + K_p\left(r(nT) - \frac{(j+0.5)S}{GR}\right)\right. \\
&\left. + K_d\left(d(nT) - \frac{(k+0.5)S}{GD}\right)\right] + K \cdot E^* \cdot R^* \cdot D^*,
\end{aligned}$$

where

$$K_p = \frac{\beta_1 \cdot GR \cdot GU \cdot H}{3L}, \qquad K_i = \frac{\beta_1 \cdot GE \cdot GU \cdot H}{3L}, \qquad K_d = \frac{\beta_1 \cdot GD \cdot GU \cdot H}{3L},$$
$$K = \frac{\beta_2 \cdot GU \cdot H}{3LS^2},$$

$$\beta_1 = \frac{2(7+\theta)S^6 - 8(1-\theta)(E^*R^* + E^*D^* + R^*D^*)S^4}{(15+\theta)S^6 - 4(1-\theta)\{16(E^*R^*D^*)^2 + 4S^2[(E^*R^*)^2 + (E^*D^*)^2 + (R^*D^*)^2] + S^4(E^{*2} + R^{*2} + D^{*2})\}}$$

and

$$\beta_2 = \frac{8(1-\theta)[3S^2 - 4(E^*R^* + E^*D^* + R^*D^*)]S^4}{(15+\theta)S^6 - 4(1-\theta)\{16(E^*R^*D^*)^2 + 4S^2[(E^*R^*)^2 + (E^*D^*)^2 + (R^*D^*)^2] + S^4(E^{*2} + R^{*2} + D^{*2})\}}.$$

Designating

$$\Delta u_G(nT) = (i+j+k+1.5)GU \cdot V,$$

$$\Delta u_{L1}(nT) = \left[K_i\left(e(nT) - \frac{(i+0.5)S}{GE}\right) + K_p\left(r(nT) - \frac{(j+0.5)S}{GR}\right) + K_d\left(d(nT) - \frac{(k+0.5)S}{GD}\right)\right],$$

and

$$\Delta u_{L2}(nT) = K \cdot E^* \cdot R^* \cdot D^*,$$

then

$$GU \cdot \Delta u(nT) = \Delta u_G(nT) + \Delta u_{L1}(nT) + \Delta u_{L2}(nT).$$

Here $\Delta u_G(nT)$ is a three-dimensional multilevel relay with respect to its inputs i, j and k. Note that

$$\Delta u_G(nT) = (i+j+k+1.5)GU \cdot V = [(i+0.5)S + (j+0.5)S + (k+0.5)S]\frac{GU \cdot H}{3L}.$$

This means that $\Delta u_G(nT)$ is calculated according to $(i+0.5)S$, $(j+0.5)S$ and $(k+0.5)S$, which are the coordinates of the center of the cube configured by $[iS, (i+1)S]$, $[jS, (j+1)S]$ and $[kS, (k+1)S]$ in which the current scaled input variables lie, with respect to the origin of the scaled input state space, (0, 0, 0). Therefore, the relay is called a global relay [23].

$\Delta u_{L1}(nT)$ is a non-linear PID controller with variable gains and its steady-state is $((i+0.5)S/GE, (j+0.5)S/GR, (k+0.5)S/GD)$, which changes with input variables. The gains, K_p, K_i and K_d vary with time because β_1 changes with, E^*, R^* and D^*. This non-linear PID controller is said to be a local controller because its control action is calculated based on E^*, R^* and D^*, which are the differences between the current scaled input variables and the center of the cube in which they lie (see (8.7)).

$\Delta u_{L2}(nT)$ is a local non-linear controller in the form of the cross product of E^*, R^* and D^*. The non-linear gain, K, also varies locally with E^*, R^* and D^*. □

The maximum control action from each of these three controllers can be determined quantitatively, as stated by the following theorem.

Theorem 2

1. $|\Delta u_G(nT)| \leq |\Delta u_G(nT)|_{\max} = \frac{N-2}{N-1} GU \cdot H;$
2. $|\Delta u_{L1}(nT) + \Delta u_{L2}(nT)| \leq |\Delta u_{L1}(nT) + \Delta u_{L2}(nT)|_{\max} = \frac{GU \cdot H}{N-1}.$ (8.9)

Proof

1. When $i = j = k = J-1$ or when $i = j = k = -J$, $|\Delta u_G(nT)|$ reaches its maximum, which is

$$|\Delta u_G(nT)|_{\max} = (3J - 1.5)GU \cdot V = \frac{N-2}{N-1}GU \cdot H.$$

Therefore,

$$|\Delta u_G(nT)| \leq |\Delta u_G(nT)|_{\max}.$$

2. When $E^* = R^* = D^* = 0.5S$ or $E^* = R^* = D^* = -0.5S$, $|\Delta u_{L1}(nT) + \Delta u_{L2}(nT)|$ reaches its maximum, which is

$$|\Delta u_{L1}(nT) + \Delta u_{L2}(nT)|_{\max} = \frac{GU \cdot H}{N-1}.$$

Hence,

$$|\Delta u_{L1}(nT) + \Delta u_{L2}(nT)| \leq \frac{GU \cdot H}{N-1}.$$ □

Theorem 2 reveals that when $N > 3$, the three-dimensional multilevel relay plays a dominant role in the fuzzy control action in comparison with the two local non-linear controllers. The larger N, the more control action from the global relay and the less from the local controllers. The work mechanism of the fuzzy controller is that the control action of the global relay is adjusted by the local non-linear controllers. The maximum amount of adjustment, however, is subject to the limitation described in (8.9). The farther the values of the scaled input variables are away from $((i+0.5), (j+0.5)S, (k+0.5)S)$, the larger the magnitude of the adjustment. When $E^* = R^* = D^* = 0.5S$ or $E^* = R^* = D^* = -0.5S$, the adjustment is maximized (which is, $GU \cdot H/(N-1)$). On the other hand, when $E^* = R^* = D^* = 0$, the adjustment is minimum (which is zero). When $N = 3$, the weight of the control action from the global and local controllers in the fuzzy control action is the same. When $N = 2$, the global relay will no longer exist and the fuzzy controller becomes a global non-linear PI controller with variable gains [25]. If N becomes larger and larger, the resolution of the relay increases, while the adjustment effect of the local controllers decreases. If N grows without bound, the fuzzy controller becomes a linear PID controller as the following theorem states (see [24] and [27] for generalized results).

Theorem 3 (Limit theorem) *When N approaches ∞,*

1. $\lim_{N\to\infty} (\Delta u_{L1}(nT) + \Delta u_{L2}(nT)) = 0,$

   and

2. $\lim_{N\to\infty} \Delta u_G(nT) = \dfrac{GU \cdot H}{3L}(GE \cdot e(nT) + GR \cdot r(nT) + GD \cdot d(nT)).$

$\lim_{N\to\infty} \Delta u_G(nT)$ (i.e. $\lim_{N\to\infty} GU \cdot \Delta u(nT)$ in this case) is a linear PID controller with the proportional gain, integral gain and derivative gain being $GR \cdot GU \cdot H/3L$, $GE \cdot GU \cdot H/3L$ and $GD \cdot GU \cdot H/3L$, respectively.

Proof

1. $\lim_{N\to\infty} |\Delta u_{L1}(nT) + \Delta u_{L2}(nT)| \leq \lim_{N\to\infty} |\Delta u_{L1}(nT) + \Delta u_{L2}(nT)|_{\max} = \lim_{N\to\infty} \dfrac{GU \cdot H}{N-1} = 0.$

   Therefore,

$$\lim_{N\to\infty} (\Delta u_{L1}(nT) + \Delta u_{L2}(nT)) = 0.$$

2. $\lim_{N\to\infty} \Delta u_G(nT) = \lim_{N\to\infty} (i+j+k+1.5)GU \cdot V = \lim_{N\to\infty} \dfrac{i+j+k}{J} \cdot \dfrac{GU \cdot H}{3}.$

Note that

$$\frac{i}{J} \le \frac{e^*}{L} \le \frac{i+1}{J}, \qquad \frac{j}{J} \le \frac{r^*}{L} \le \frac{j+1}{J} \quad \text{and} \quad \frac{k}{J} \le \frac{d^*}{L} \le \frac{k+1}{J}.$$

Hence,

$$\lim_{N\to\infty} \frac{i}{J} = \frac{e^*}{L}, \quad \lim_{N\to\infty} \frac{j}{J} = \frac{r^*}{L} \quad \text{and} \quad \lim_{N\to\infty} \frac{k}{J} = \frac{d^*}{L}.$$

As a result,

$$\lim_{N\to\infty} \Delta u_G(nT) = \frac{GU \cdot H}{3L}(GE \cdot e(nT) + GR \cdot r(nT) + GD \cdot d(nT)). \qquad \square$$

A practical implication of Theorem 3 is that when linear fuzzy control rules are adopted, the fuzzy controller designer should select a relatively small number of input fuzzy sets in order to take advantage of the non-linearity of the fuzzy controller. The more input fuzzy sets are used, the less non-linear the fuzzy controller will be.

In our previous work we have studied some fuzzy controllers that use two input variables, error and rate, and linear fuzzy control rules. Our analytically derived results have shown that these fuzzy controllers are the sum of a global two-dimensional multilevel relay and a local non-linear PI controller (e.g. [23], [26]). Note that the configurations of these fuzzy controllers are different from those in this chapter. Nevertheless, the conclusion on the controller structure is applicable to the fuzzy controller in the present chapter when it uses only two input variables, as stated by the following theorem.

Theorem 4 *When the fuzzy controller uses error and rate as input variables instead of the three input variables, its structure becomes the sum of a global two-dimensional multilevel relay and a local non-linear PI controller.*

Proof The proof of this theorem is similar to that of Theorem 1. When d_rate is excluded from the input variables, the total number of output fuzzy sets reduces to $4J + 1$ (i.e. $2N - 1$) and the space between the central values of two adjacent output fuzzy sets becomes

$$V = \frac{H}{2J}.$$

Also, only N^2 linear control rules will be needed:

If e^* is E_i AND r^* is R_j **Then** $\Delta u(nT)$ is ΔU_{i+j}.

The structure of the fuzzy controller is derived as follows:

$$GU \cdot \Delta u(nT) = (i+j+1)GU \cdot V + \left[K_i\left(e(nT) - \frac{i+0.5)S}{GE}\right) + K_p\left(r(nT) - \frac{j+0.5)S}{GR}\right)\right],$$

where

$$K_p = \frac{\beta_1 \cdot GR \cdot GU \cdot H}{2L}; \qquad K_i = \frac{\beta_1 \cdot GE \cdot GU \cdot H}{2L},$$

and

$$\beta_1 = \frac{2(3+\theta)S^4 - 8(1-\theta)E^*R^*S^2}{(7+\theta)S^4 - 4(1-\theta)\{(E^*S)^2 + 4(E^*R^*)^2 + (R^*S)^2\}}.$$

Obviously, $GU \cdot \Delta u(nT)$ consists of a global two-dimensional multilevel relay and a local non-linear PI controller. □

It can easily be proved that the absolute value of the maximal outputs of the global two-dimensional relay and the local non-linear PI controller are $(N-2)GU \cdot H/(N-1)$ and $GU \cdot H/(N-1)$, respectively. As the theorem below states, the global two-dimensional multilevel relay approaches a global linear PI controller as $N \to \infty$, whereas the local non-linear PI controller disappears (see also [1], [2], [23], [24] for related results].

Theorem 5 *If the fuzzy controller uses error and rate as input variables, then*

$$\lim_{N \to \infty} GU \cdot \Delta u(nT) = \frac{GU \cdot H}{2L}(GE \cdot e(nT) + GR \cdot r(nT)).$$

Proof See the proof of Theorem 3.

8.4 CONCLUSION

The explicit structure of the fuzzy controller using error, rate and d_rate of process output as input variables has been derived. The result is the sum of a global three-dimensional multilevel relay and two local non-linear controllers. One of the local non-linear controllers is a non-linear PID controller with variable proportional gain, integral gain and derivative gain. We have shown that, when $N > 3$, the major portion of the control action of the fuzzy controller is contributed by the global relay while the local non-linear controllers merely fine tune the output of the global controller. The role of the global and local controllers in the fuzzy control action have been quantitatively determined. Additionally, it has been proved that the limit structure of the fuzzy controller is determined by the global multilevel relay which becomes a global linear PID controller as the number of input fuzzy sets grows without bound. In the meantime, the two local non-linear controllers disappear.

As we have shown in our previous paper [23], the structure of the fuzzy controller using linear control rules and error and rate as input variables is the sum of a global two-dimensional multilevel relay and a local non-linear PI controller. Logically, one would expect that when the additional input variable d_rate is involved, the fuzzy controller structure should be the sum of a global three-dimensional multilevel relay and a local non-linear PID controller. We show that this is not the case for the fuzzy controller studied in this chapter. Whether this conjecture holds for other fuzzy controllers that use the three input variables remains a future research topic.

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