Feedback-linearization-based control of discrete-time quadratic TS fuzzy systems with disturbances

Liwei Ren\textsuperscript{a}, Xiaojun Ban\textsuperscript{a} and Hao Ying\textsuperscript{b,∗}
\textsuperscript{a}Center for Control Theory and Guidance Technology, Harbin Institute of Technology, Harbin, China
\textsuperscript{b}Department of Electrical and Computer Engineering, Wayne State University, Detroit, Michigan, USA

Abstract. Controlling a dynamic system to make its output identical to a user-desired reference trajectory given to the system’s input without any delay (i.e., perfect output tracking control) is an important and frequently encountered control requirement in industries (e.g., robotic control). When disturbances exist in a system, perfect output tracking control may be impossible to attain and alternatively asymptotical output tracking is sought. This paper presents an asymptotical output tracking control design method for a general class of discrete-time TS fuzzy systems with quadratic rule consequents, which offer better modeling capabilities than the linear rule consequents. The control problem is dealt with by utilizing the feedback linearization method. To guarantee asymptotical output tracking performance in the presence of square disturbance signal, an auxiliary PI controller is added to attenuate the disturbance. The feedback linearization method is known in the literature to fail to work for certain systems because it can make the tracking controller’s output unbounded. To address this issue, we put forward a full block S-procedure condition to check whether such failure will occur for any given quadratic TS fuzzy system. Applying feedback linearization to the quadratic TS fuzzy systems is innovative relative to the literature that has exclusively dealt with the TS fuzzy systems with linear rule consequents only. Two numerical examples are provided to illustrate the effectiveness and utility of our theoretical results.

Keywords: Discrete-time quadratic TS fuzzy systems, asymptotical output tracking, disturbance rejection, feedback linearization, full block S-procedure

1. Introduction

Feedback linearization is a well-established nonlinear controller design technique [1, 2] whose central idea is to use the feedback to cancel internal nonlinearities of the system to be controlled to make the closed-loop control system linear so that linear control techniques can be applied. In addition to theoretical developments [3–5], the technique has been successfully used in practical applications. These include the control of the quadrotor helicopter [6], the robot manipulator [7], the nonholonomic-wheeled mobile robot [8], and the hypersonic vehicle [9].

In a previous paper [10], we applied the feedback linearization control approach to fuzzy systems. We established a feedback-linearization-based framework for systematically designing an output tracking controller for a general class of the discrete-time Takagi-Sugeno (TS) fuzzy systems with linear rule consequent so that the output of the controlled fuzzy system perfectly tracked a user-desired bounded reference trajectory. The fuzzy system under control was allowed to be unstable. Necessary and sufficient conditions were derived to check its stability as well as
the stability of the designed tracking controller. In this paper, we extend this feedback-linearization-based framework to cover a new and broader general class of discrete-time fuzzy systems, which are the TS fuzzy systems whose rule consequents are quadratic (i.e., second-order polynomials of the input variables).

Quadratic rule consequent, and more generally polynomial rule consequent, has attracted attention of fuzzy control researchers in the recent years because they offer better modeling capabilities than the linear rule consequent. For instance, system analysis and controller design has been conducted using the Sum-of-Squares (SOS) optimization approach for the TS fuzzy systems with polynomial rule consequent [11–20]. It is important to point out that various non-feedback-linearization control schemes, including the SOS-based approach, have been developed in the literature to attain output tracking control (but not perfect output tracking control). Hence, these schemes and their related studies are really not relevant. From now on, we will focus on the fuzzy feedback linearization control scheme and literature only.

There exist only a handful of publications on feedback linearization control of a fuzzy system besides our own early work in [10]. In [21, 22], a $L_2$ robust stability analysis and design method was proposed for the feedback linearization regulator of the TS fuzzy systems via multivariable circle criterion. In [23, 24], a linear-matrix-inequality-based robust stability condition, which could be solved numerically, was developed for the feedback linearization control of the TS fuzzy systems. In [25], a study on the relative degrees and adaptive tracking control of the single-input, single-output TS fuzzy systems in general non-canonical forms was presented. Applications of the feedback linearization method to the balance control of the inverted pendulum systems and to the regulation control of a single-link flexible joint manipulator were reported respectively in [26, 27]. Our comprehensive literature search indicates that only above eight reports in the entire literature that involve fuzzy feedback linearization and hence they are the only ones relevant. All the fuzzy systems used linear rule consequents.

While the feedback linearization approach is powerful for certain systems, it suffers some limitations. First, it cannot handle disturbances effectively. Second, it cannot be applied to all nonlinear systems. The disturbance issue was tackled only in [21, 22] and their approaches were graphical and for the TS fuzzy systems with linear rule consequents. When disturbances exist, obtaining the perfect output tracking control may be impossible. For the second problem, the literature recognizes that under certain fuzzy-system-dependent conditions, the designed controllers fail because the controllers’ output can approach infinity [21–27]. No effort, including our work [10], has been reported on how to determine which fuzzy systems will not cause such failure.

The objectives of the present paper were to address these problems by: (1) incorporating the PI controller design approach into the feedback linearization method to obtain the asymptotical output tracking in the presence of disturbances, and (2) utilizing the full block S-procedure method to develop a criterion that is related to linear-matrix-inequality (LMI) [28, 29] for the quadratic TS fuzzy systems.

There are three aspects of novelty in our work: (1) it is the first attempt at applying the feedback linearization to the discrete-time quadratic TS fuzzy system; (2) the PI controller has been added to the feedback linearization controller to guarantee asymptotical output tracking performance in the presence of disturbances; (3) the established condition can be used to inform that some quadratic TS fuzzy systems are not suitable for the feedback linearization controller.

The rest of this paper is organized as follows. The TS fuzzy systems covered in this paper are presented in Section 2. In Section 3, we present our asymptotical output tracking controller design and utilize the full block S-procedure method to establish a sufficient condition for determining if a given fuzzy system will not cause the designed controller to fail. In Section 4, two numerical examples are presented to illustrate the validity of our theoretical results. Section 5 concludes the paper.

2. Discrete-time quadratic TS fuzzy systems

Discrete-time quadratic TS fuzzy systems to be controlled via feedback linearization are assumed to have the following configuration. They have a total of $\Omega$ rules and the $\ell$-th rule is

\[
R^{\ell}: \text{IF } y(n) \text{ is } M^{\ell}_0 \text{ AND } y(n-1) \text{ is } M^{\ell}_1 \text{ AND } \ldots \text{ AND } y(n-m) \text{ is } M^{\ell}_m, \text{ THEN }
\]

\[
y(n + 1) = f^{\ell}(\tilde{y}(n)) + \sum_{k=0}^{p} g^{\ell}_k(\tilde{y}(n)) u(n - k) + d(n)
\]

(1)
where \( y(n) \) and \( u(n) \) are the system output and input at sampling instant \( n \), respectively. \( d(n) \) is the disturbance signal, which is assumed to be a square one. \( a^\ell, \ b^\ell, q^\ell, v^\ell \) and \( s^\ell \) are constants with \( q^\ell \neq 0 \). \( M_i^\ell \) is a fuzzy set fuzzifying \( y(n-i) \) and its membership function is denoted \( \mu_i^\ell(y(n-i)) \). Fuzzy logic AND operators in the rules can be any types and different AND operators may be used in a rule. We use the symbol \( \otimes \) to represent the AND operators. The combined degree of membership for the \( \ell \)-th rule is

\[
\mu_i^\ell(\vec{y}(n)) = \mu_i^0(y(n)) \otimes \mu_1^\ell(y(n-1)) \otimes \ldots \otimes \mu_m^\ell(y(n-m)),
\]

where \( \vec{y}(n) \triangleq (y(n), y(n-1), \ldots, y(n-m)) \). Obviously, \( \mu_i^\ell(\vec{y}(n)) \geq 0, \ \ell = 1, 2, \ldots, \Omega \) and it is required that

\[
\sum_{\ell=1}^{\Omega} \mu_i^\ell(\vec{y}(n)) > 0,
\]

which is commonly satisfied by the well-developed fuzzy systems. The generalized defuzzifier [30] is used to produce a crisp output for the systems:

\[
y(n+1) = \frac{\sum_{\ell=1}^{\Omega} (\mu_i^\ell(\vec{y}(n)))^\alpha \left( f^\ell(\vec{y}(n)) + \sum_{k=0}^{p} g^\ell_k(\vec{y}(n)) u(n-k) \right)}{\sum_{\ell=1}^{\Omega} (\mu_i^\ell(\vec{y}(n)))^\alpha} + d(n)
\]

\[
= \sum_{\ell=1}^{\Omega} h^\ell_1(\vec{y}(n)) \left( f^\ell(\vec{y}(n)) + \sum_{k=0}^{p} g^\ell_k(\vec{y}(n)) u(n-k) \right)
\]

\[
+ d(n),
\]

where

\[
h^\ell_1(\vec{y}(n)) = \left( \frac{\sum_{\ell=1}^{\Omega} (\mu_i^\ell(\vec{y}(n)))^\alpha \left( f^\ell(\vec{y}(n)) + \sum_{k=0}^{p} g^\ell_k(\vec{y}(n)) u(n-k) \right)}{\sum_{\ell=1}^{\Omega} (\mu_i^\ell(\vec{y}(n)))^\alpha} \right)^\frac{1}{\alpha} \geq 0
\]

and

\[
\sum_{\ell=1}^{\Omega} h^\ell(\vec{y}(n)) = 1.
\]

Here, \( \alpha \ (0 \leq \alpha \leq +\infty) \) is a design parameter. Different values of \( \alpha \) produce different defuzzification strategies. The popular centroid defuzzifier and the mean of maximum defuzzifier are just two special cases when \( \alpha = 1 \) and \( \alpha = \infty \), respectively [30].

### 3. Asymptotical output tracking controller design and the condition for its existence

This section describes the asymptotical output tracking controller design when the square disturbance signal exists as well as a sufficient condition for determining which fuzzy systems will not cause the controller to fail (i.e., existence of the controller). We show the incorporation of the PI controller with the feedback linearization controller and the utilization of the full block \( S \)-procedure method.

#### 3.1. Asymptotical output tracking controller design

For a given signal sequence \( \{y_r(n)\} \), the asymptotical output tracking problem is to find an input sequence \( \{u(n)\} \) that will make the output \( y(n) \) of the closed-loop system satisfy the following condition

\[
\lim_{n \to \infty} [y(n) - y_r(n)] = 0,
\]

where the tracked signal \( y_r(n) \) is called the reference signal. We consider the case that \( y_r(n) = y_r \) is a step signal in our work, and the condition (5) becomes

\[
\lim_{n \to \infty} y(n) = y_r.
\]

Rewriting the fuzzy systems (2) to another form:

\[
y(n+1) = \sum_{\ell=1}^{\Omega} h^\ell_1(\vec{y}(n)) \left( f^\ell(\vec{y}(n)) + \sum_{k=0}^{p} g^\ell_k(\vec{y}(n)) u(n-k) \right)
\]

\[
+ \sum_{\ell=1}^{\Omega} h^\ell_1(0) \left( \sum_{i=0}^{m} a^\ell_i y(n-i) \right) + d(n)
\]

\[
= \sum_{\ell=1}^{\Omega} h^\ell_1(0) \left( \sum_{i=0}^{m} a^\ell_i y(n-i) \right) + \sum_{\ell=1}^{\Omega} h^\ell_1(\vec{y}(n)) \left( f^\ell(\vec{y}(n)) + \sum_{k=0}^{p} g^\ell_k(\vec{y}(n)) u(n-k) \right)
\]

\[
+ \sum_{\ell=1}^{\Omega} h^\ell_1(0) \left( \sum_{i=0}^{m} a^\ell_i y(n-i) \right) + d(n)
\]

\[
- \sum_{\ell=1}^{\Omega} h^\ell_1(0) \left( \sum_{i=0}^{m} a^\ell_i y(n-i) \right) + d(n).
\]
We design controller with an auxiliary input $v(n)$ as following:

$$u(n) = \frac{v(n) + \sum_{\ell=1}^{\Omega} h_\ell(0) \left( \sum_{i=0}^{m} a_i^\ell y(n-i) \right)}{\sum_{\ell=1}^{\Omega} h_\ell(\tilde{y}(n)) g_{0,\ell}(\tilde{y}(n))} - \sum_{\ell=1}^{\Omega} h_\ell(\tilde{y}(n)) f(\tilde{y}(n)) + \sum_{k=1}^{p} a_k^\ell(\tilde{y}(n)) u(n-k)} \right)$$

(assuming $\sum_{\ell=1}^{\Omega} h_\ell(\tilde{y}(n)) g_{0,\ell}(\tilde{y}(n)) \neq 0$ for all $n$. We will address this assumption later). Substituting the controllers (6) into the fuzzy systems (2) produces the closed-loop control system

$$y(n + 1) = \sum_{\ell=1}^{\Omega} h_\ell(0) \left( \sum_{i=0}^{m} a_i^\ell y(n-i) \right) + v(n) + d(n) \quad (7)$$

By defining

$$\hat{a}_i \triangleq \sum_{\ell=1}^{\Omega} h_\ell(0) a_i^\ell \quad (8)$$

Equation (7) becomes

$$y(n + 1) = \sum_{i=0}^{m} \hat{a}_i y(n-i) + v(n) + d(n) \quad (9)$$

Selecting state variables

$$x_1(n) \triangleq y(n-m)$$

$$x_2(n) \triangleq y(n-m+1)$$

$$\vdots$$

$$x_{m+1}(n) \triangleq y(n),$$

and output variable

$$z(n) \triangleq y(n),$$

then system (9) can be transformed to the following state space form

$$\begin{cases}
\tilde{x}(n + 1) = A\tilde{x}(n) + B v(n) + B d(n) \\
z(n) = C\tilde{x}(n)
\end{cases} \quad (10)$$

where $\tilde{x}(n) = [x_1(n), x_2(n), \ldots, x_{m+1}(n)]^T$, and

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix} \in \mathbb{R}^{(m+1) \times (m+1)},$$

$$B = [0, \ldots, 0, 1]^T \in \mathbb{R}^{(m+1) \times 1},$$

$$C = [0, \ldots, 0, 1] \in \mathbb{R}^{1 \times (m+1)}. \quad (11)$$

Defining the error signal

$$e(n) \triangleq y(n) - y_r, \quad (12)$$

and introducing an auxiliary state variable $\xi(n)$ as follows:

$$\xi(n) = \sum_{i=0}^{n-1} (y(i) - y_r),$$

which produces

$$\xi(n + 1) = y(n) - y_r + \xi(n),$$

leading to

$$e(n) = \xi(n + 1) - \xi(n). \quad (13)$$

By putting Equations (10 and 13) together, the augmented system is obtained:

$$\begin{bmatrix} \tilde{x}(n + 1) \\ \xi(n + 1) \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}(n) \\ \xi(n) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v(n)$$

$$- \begin{bmatrix} 0 \\ 1 \end{bmatrix} y_r + \begin{bmatrix} B \\ 0 \end{bmatrix} d(n),$$

$$z(n) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}(n) \\ \xi(n) \end{bmatrix}. \quad (14)$$
Inspired by [31], we choose the state-feedback controller for (14) as follows:

\[
v(n) = \ddot{K} \begin{bmatrix} \ddot{x}(n) \\ \xi(n) \end{bmatrix} = \begin{bmatrix} K_x, & K_q \end{bmatrix} \begin{bmatrix} \ddot{x}(n) \\ \xi(n) \end{bmatrix}
\]

\[
= \begin{bmatrix} K_m, & K_{m-1}, \ldots, & K_0 \end{bmatrix} \begin{bmatrix} y(n-m) \\ y(n-m+1) \\ \vdots \\ y(n) \end{bmatrix} + K_q \sum_{i=0}^{n-1} (y(i) - y_r),
\]

where \( \ddot{K} = [\ddot{K}_x, \ddot{K}_q] = [K_m, K_{m-1}, \ldots, K_0, K_q] \) is the controller gain vector, which is determined by the desired pole placement for the closed-loop system.

With the development above, the controller (6) can be expressed as

\[
u(n) = \frac{\sum_{i=0}^{m} K_i y(n-i) + K_q \sum_{i=0}^{n-1} (y(i) - y_r) + \sum_{i=0}^{m} K_{i} \xi(n-i)}{\sum_{i=1}^{\Omega} h_i(\ddot{y}(n)) g_i(\ddot{y}(n)) - \sum_{i=1}^{\Omega} h_i(\ddot{y}(n)) g_i(\ddot{y}(n))}
\]

(15)

\[
= \begin{bmatrix} \ddot{x}(n) \\ \xi(n) \end{bmatrix} \begin{bmatrix} y(n-m) \\ y(n-m+1) \\ \vdots \\ y(n) \end{bmatrix} + K_q \sum_{i=0}^{n-1} (y(i) - y_r),
\]

Note that the signals needed in controller (16) are all measurable. Therefore, the controller can be implemented.

**Theorem 1.** If (15) is a state-feedback stabilization controller of a augmented system (14), then the output of fuzzy systems (2) can asymptotically track the step reference signal with zero steady-state error under controller (16) if its denominator \( \sum_{\ell=1}^{\Omega} h_\ell(\ddot{y}(n)) g_\ell(\ddot{y}(n)) \neq 0 \) for all \( n \).

**Proof.** System (2) under controller (16) forms a closed-loop system, which is equivalent to the closed-loop system formed by system (14) controlled by controller (15):

\[
\ddot{x}(n + 1) = \mathcal{A}_{cl} \ddot{x}(n) - \begin{bmatrix} 0 & B \end{bmatrix} y_r + \begin{bmatrix} 0 & B \end{bmatrix} d(n),
\]

(17)

where

\[
\ddot{x}(n) = \begin{bmatrix} \ddot{x}(n) \\ \ddot{y}(n) \end{bmatrix}, \quad \mathcal{A}_{cl} = \begin{bmatrix} A + B \ddot{K}_x & B K_q \\ C & 1 \end{bmatrix}.
\]

In (1), we assume that the disturbance signal is a square one. So from (17), one gets

\[
\ddot{x}(n+1) - \ddot{x}(n-1) = \mathcal{A}_{cl} \left( \ddot{x}(n) - \ddot{x}(n-1) \right).
\]

With the controller (15) being a state feedback stabilization controller, system (17) is asymptotically stable. This means all the eigenvalues of \( \mathcal{A}_{cl} \) are inside the unit circle and the state of the system satisfies

\[
\ddot{x}(n) - \ddot{x}(n-1) \to 0, \quad n \to \infty.
\]

From Equation (13), we can obtain

\[
e(n) = \ddot{x}(n+1) - \ddot{x}(n) \to 0, \quad n \to \infty,
\]

and accordingly,

\[
\lim_{n \to \infty} y(n) = y_r.
\]

i.e., the output of fuzzy systems (2) can asymptotically track the step reference signal. \( \blacksquare \)

Theorem 1 tells us that the problem of designing a step reference signal tracking controller for fuzzy systems (2) can be converted to the state feedback stabilization problem of designing augmented system (14). How to solve the latter problem? The following theorem gives the answer.

**Theorem 2.** If \( (A, B) \) is controllable, the augmented system (14) can be completely controlled if and only if

\[
\begin{bmatrix} 1 - A & B \\ -C & 0 \end{bmatrix} = m + 2,
\]

(18)

where \( A \in \mathbb{R}^{(m+1) \times (m+1)}, B \in \mathbb{R}^{(m+1) \times 1} \) and \( C \in \mathbb{R}^{1 \times (m+1)} \).

**Proof.** For augmented system (14), its controllability is equivalent to

\[
\begin{bmatrix} \lambda_i I - \begin{bmatrix} A & 0 \\ C & 1 \end{bmatrix} B \end{bmatrix} = m + 2, \quad i = 1, 2, \ldots, m + 2,
\]

where the Allerton Press watermark is removed.
where \( \lambda_i (i = 1, 2, \ldots, m+2) \) are roots of
\[
\begin{vmatrix}
\lambda I - \begin{bmatrix} A & 0 \\ C & 1 \end{bmatrix}
\end{vmatrix} = \begin{vmatrix}
\lambda I - A & 0 \\ -C & \lambda - 1
\end{vmatrix} = |\lambda I - A| (\lambda - 1) = 0,
\]
In other words, \( \lambda_i = 1 \) and \( |\lambda I - A| = 0 \).

(1) When \( \lambda_i = 1 \)
\[
\operatorname{rank} \left[ \lambda_i I - \begin{bmatrix} A & 0 \\ C & 1 \end{bmatrix} \right] = \operatorname{rank} \left[ \begin{bmatrix} I - A & 0 \\ -C & 0 \end{bmatrix} \right] = \operatorname{rank} \left[ \begin{bmatrix} I - A & \beta \\ -C & 0 \end{bmatrix} \right] = m + 2.
\]

(2) When \( \lambda_i \) are the roots of \( |\lambda I - A| = 0 \) and \( \lambda_i \neq 1 \)

As \((A, \beta)\) is controllable, we can obtain
\[
\operatorname{rank} \left[ \lambda_i I - \begin{bmatrix} A & 0 \\ C & 1 \end{bmatrix} \right] = m + 1, \quad i = 1, 2, \ldots, m + 1,
\]
and accordingly
\[
\operatorname{rank} \left[ \lambda_i I - \begin{bmatrix} A & 0 \\ C & 1 \end{bmatrix} \right] = \operatorname{rank} \left[ \begin{bmatrix} \lambda_i I - A & 0 \\ -C & \lambda_i - 1 \end{bmatrix} \right] = \operatorname{rank} \left[ \lambda_i I - A \right] + \operatorname{rank} [\lambda_i - 1] = (m + 1) + 1 = m + 2,
\]
which concludes the proof.

Therefore, when \((A, \beta)\) is controllable and Equation (18) holds, the closed-loop system produced by fuzzy systems (2) and controller (16) can place the poles arbitrarily.

Theorem 1 requires that
\[
\sum_{\ell=1}^{\Omega} h \ell (\tilde{y}(n)) g_0^\ell \tilde{y}(n) \neq 0 \text{ for all } n. \]
One may wonder whether it is possible to determine prior to the operation of the control system whether the condition is met because it depends on \( \tilde{y}(n) \) which is unknown beforehand? Our answer is affirmative and a sufficient condition has been established in the next section.

### 3.2. A sufficient condition for determining the controller existence based on the full block S-procedure method

**Lemma 1.** *(Full block S-procedure method [32]):* For a given linear fractional transformation (LFT) system
\[
G(\Delta) = \Delta \ast \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}
\]
and a quadratic matrix inequality
\[
G^T(\Delta) MG(\Delta) < 0
\]
with \( \Delta \in \mathbb{D} \) and \( \{0\} \subset \mathbb{D} \). Inequality (20) holds if and only if there exists a full-block multiplier \( \Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{T_1}^T & \Pi_{22} \end{bmatrix} \) such that
\[
\begin{bmatrix} G_{11} & G_{12} \\ I & 0 \\ G_{21} & G_{22} \end{bmatrix}^T \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{T_1}^T & \Pi_{22} \end{bmatrix} \begin{bmatrix} 0 & G_{11} & G_{12} \\ I & 0 \\ 0 & 0 & M \end{bmatrix} < 0,
\]
and for any \( \Delta \in \mathbb{D} \)
\[
\begin{bmatrix} I^T \phi_1 \Pi_{12} \\ \phi_2 \Pi_{22} \\ \Delta \end{bmatrix} \geq 0. \quad (21)
\]

Note that inequality constraint (21) consists of the quadratic term of matrix \( \Delta \). In general, this constraint is a non-convex constraint. Even if \( \mathbb{D} \) is a convex set, it needs to check an infinite number of inequality constraints to make sure inequality constraint (21) holds. By introducing an additional constraint \( \Pi_{22} \leq 0 \), (21) becomes a convex constraint associated with \( \Delta \).

Therefore, when \( \mathbb{D} \) is a convex polyhedron set, it only needs to verify inequality (21) at a finite number of vertices.

**Theorem 3.** For given scalars \( \phi < 0, \bar{\phi} > 0 \), if
\[
\text{there exists a symmetric matrix } \Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{T_1}^T & \Pi_{22} \end{bmatrix}
\]
with \( \Pi_{11} \in \mathbb{R}^{1 \times 1}, \Pi_{12} \in \mathbb{R}^{1 \times (m+1)}, \text{ and } \Pi_{22} \in \mathbb{R}^{(m+1) \times (m+1)} \text{ that meets the following } \Omega + 2m^2 + 1 \text{ inequality constraints:}
\[ \Pi + \left( \mathcal{H}^{(v)} \right)^T \tilde{\Theta} < 0, \quad v = 1, 2, \ldots, \Omega \]  
(22)

(or change \( \tilde{\Theta} \) to \( -\tilde{\Theta} \) in inequality (22)), and

\[ \begin{bmatrix} I \\ \Delta^{(\sigma)} \end{bmatrix}^T \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^T & \Pi_{22} \end{bmatrix} \begin{bmatrix} I \\ \Delta^{(\sigma)} \end{bmatrix} \geq 0, \quad \sigma = 1, 2, \ldots, 2^{m+1} \]  
(23)

\[ \Pi_{22} \leq 0, \quad \Pi_{22} \leq 0, \]  
(24)

the following formula can be obtained:

\[ \sum_{\ell=1}^{\Omega} h_\ell (\tilde{y}(n)) g_\ell^0 (\tilde{y}(n)) \neq 0 \]  
(25)

with

\[ \phi \leq y(n-i) \leq \bar{\phi} \left( \phi < 0, \bar{\phi} > 0 \right), \]  
(26)

\[ i = 0, 1, \ldots, m \]

where

\[ \Delta = [y(n), y(n-1), \ldots, y(n-m)]^T, \]  
(27)

\( (\Delta^{(\sigma)} \) is determined by inequality (26) and Equation (27))

\[ \begin{align*}
G_{11} &= 0_{1 \times (m+1)} \\
G_{12} &= I_1 \\
G_{21} &= \begin{bmatrix} 0_{1 \times (m+1)} \\ I_{m+1} \\ \vdots \\ 0_{(m+2) \times 1} \end{bmatrix}, \\
G_{22} &= \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0_{(m+2) \times 1} \end{bmatrix}
\end{align*} \]  
(28)

and

\[ \mathcal{H}^{(v)}, \quad v = 1, 2, \ldots, \Omega \]  
(29)

with

\[ \mathcal{H}^{(1)} = \begin{bmatrix} I_{m+2} \\ 0_{m+2} \\ \vdots \\ 0_{m+2} \end{bmatrix}, \quad \mathcal{H}^{(2)} = \begin{bmatrix} 0_{m+2} \\ I_{m+2} \\ \vdots \\ 0_{m+2} \end{bmatrix}, \]  

(30)

and

\[ \tilde{\Theta} = [\tilde{\Theta}_1, \tilde{\Theta}_2, \ldots, \tilde{\Theta}_\Omega]^T \]  
(31)

where

\[ \begin{align*}
\tilde{\Theta}_i &= \begin{bmatrix} \tilde{A}_i \\ \tilde{B}_i \\ \tilde{C}_i \end{bmatrix}, \quad i = 1, 2, \ldots, \Omega \\
\tilde{A}_i &= q_0^{i,0}, \\
\tilde{B}_i &= 0.5 \left[ v_0^{0,0}, v_1^{0,0}, \ldots, v_m^{0,0} \right], \\
\tilde{C}_i^{j,k} &= \ell_{j,k}^{i,0}, \quad 0 \leq j \leq m, \quad 0 \leq k \leq m \\
\ell_{j,k}^{i,0} &= \min \left( \frac{\omega_{j+k}^{(i)}}{\omega_{j+k}^{(i)}}, \alpha_{j+k}^{(i)} \right) \\
\omega_{j+k}^{(i)} &= \max \left( \omega_{j+k}^{(i)}, \alpha_{j+k}^{(i)} \right)
\end{align*} \]  

Proof. From (25), we have

\[ \sum_{\ell=1}^{\Omega} h_\ell (\tilde{y}(n)) g_\ell^0 (\tilde{y}(n)) = \sum_{\ell=1}^{\Omega} h_\ell (\tilde{y}(n)) \left( q_0^{\ell,0} + \sum_{i=0}^{m} v_i^{\ell,0} y(n-i)y(n-j) \right) \neq 0 \]  

which means

\[ \sum_{\ell=1}^{\Omega} h_\ell (\tilde{y}(n)) \left( q_0^{\ell,0} + \sum_{i=0}^{m} v_i^{\ell,0} y(n-i) \right) + \sum_{j=0}^{m} \sum_{i=0}^{m} \ell_{j,k}^{i,0} y(n-i)y(n-j) < 0 \]  
(32)

or

\[ \sum_{\ell=1}^{\Omega} h_\ell (\tilde{y}(n)) \left( q_0^{\ell,0} - \sum_{i=0}^{m} v_i^{\ell,0} y(n-i) \right) + \sum_{j=0}^{m} \sum_{i=0}^{m} \ell_{j,k}^{i,0} y(n-i)y(n-j) > 0. \]  
(33)
Starting with inequality (32) with $\tilde{\Theta}$ in (22) (inequality (33) is similar with $-\tilde{\Theta}$), and expressing the left side of inequality (32) in a matrix form produces
\[
\sum_{\ell=1}^{\Omega} h_\ell (\tilde{y} (n)) \left( q_0^{\ell,0} + \sum_{i=0}^{m} v_i^{\ell,0} y(n-i) + \sum_{i=0}^{m} \sum_{j=i}^{m} s_{i,j}^{\ell,0} y(n-i) y(n-j) \right) = Y^T \Psi Y < 0,
\]
where
\[
Y = \begin{bmatrix} 1, y(n), y(n-1), \ldots, y(n-m) \end{bmatrix}^T
\]
and
\[
\Psi = \begin{bmatrix} D & E \\ E^T & F \end{bmatrix}
\]
with
\[
D = \sum_{\ell=1}^{\Omega} h_\ell (\tilde{y} (n)) q_0^{\ell,0},
\]
\[
E = 0.5 \begin{bmatrix} \sum_{\ell=1}^{\Omega} h_\ell (\tilde{y} (n)) q_0^{\ell,0} & 0 & \cdots & 0 \\ 0 & \sum_{\ell=1}^{\Omega} h_\ell (\tilde{y} (n)) q_0^{\ell,0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{\ell=1}^{\Omega} h_\ell (\tilde{y} (n)) q_0^{\ell,0} \end{bmatrix},
\]
\[
F = \begin{bmatrix} F_{1,1} & F_{1,2} & \cdots & F_{1,m+1} \\ F_{2,1} & F_{2,2} & \cdots & F_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots \\ F_{m+1,1} & F_{m+1,2} & \cdots & F_{m+1,m+1} \end{bmatrix},
\]
\[
F_{i,i} = \sum_{\ell=1}^{\Omega} h_\ell (\tilde{y} (n)) s_{i-1,i-1}^{\ell,0}, \quad 1 \leq i \leq m + 1,
\]
\[
F_{i,j} = 0.5 \sum_{\ell=1}^{\Omega} h_\ell (\tilde{y} (n)) s_{i-1,j-1}^{\ell,0}, \quad 1 \leq i < j \leq m + 1,
\]
\[
F_{i,j} = F_{j,i}^T, \quad 1 \leq i < j \leq m + 1.
\]

Rewriting $G (\Delta)$ in (19) produces
\[
G (\Delta) = \Delta \ast \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}
\]
\[
= G_{22} + G_{21} \Delta (I - G_{11} \Delta)^{-1} G_{12}.
\]

Let $G_{11} = 0$ and we can get
\[
G (\Delta) = G_{22} + G_{21} \Delta G_{12}. \tag{35}
\]

Now we transform the vector $Y$ to LFT form (35):
\[
Y = \begin{bmatrix} y(n) \\ y(n-1) \\ \vdots \\ y(n-m) \end{bmatrix} + \begin{bmatrix} 0_{1 \times (m+1)} \\ I_{m+1} \end{bmatrix} \cdot I_1,
\]
and $\Delta, G_{11}, G_{12}, G_{21}$ and $G_{22}$ are as shown in Equations (27 and 28).

By utilizing Lemma 1, for any $\Delta \in \mathcal{D}$, if there exists a matrix $\Pi$ meeting the following two inequality constraints
\[
\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}^T \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^T & \Pi_{22} \end{bmatrix} \begin{bmatrix} G_{11} \\ G_{12} \end{bmatrix} < 0, \tag{36}
\]
\[
\begin{bmatrix} I^T \\ \Delta \end{bmatrix} \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^T & \Pi_{22} \end{bmatrix} \begin{bmatrix} I \\ \Delta \end{bmatrix} \geq 0, \tag{37}
\]
then we can obtain inequality (32)
\[
\sum_{\ell=1}^{\Omega} h_\ell (\tilde{y} (n)) \left( q_0^{\ell,0} + \sum_{i=0}^{m} v_i^{\ell,0} y(n-i) + \sum_{i=0}^{m} \sum_{j=i}^{m} s_{i,j}^{\ell,0} y(n-i) y(n-j) \right) < 0.
\]

Furthermore, multiplying the left side of inequality (36) with $G_{11}, G_{12}, G_{21}$ and $G_{22}$, we can obtain its equivalent inequality
\[
\Pi + \Psi < 0. \tag{38}
\]
Rewriting $\Psi$ in (34) can produce

$$\Psi = \sum_{\ell = 1}^{\Omega} h_\ell (\gamma(n)) = H^T \Theta,$$  

(39)

where

$$H = [h_1 (\gamma(n)) \, I_{m+2}, \, h_2 (\gamma(n)) \, I_{m+2}, \ldots, \, h_\Omega (\gamma(n)) \, I_{m+2}]^T,$$

and $\Theta$ is shown in Equation (30).

The set

$$\mathbb{H} = \left\{ H \in \mathbb{R}^{(m + 2)\Omega \times (m + 2)} \left| 0 \leq h_\ell (\gamma(n)) \leq 1, \sum_{\ell = 1}^{\Omega} h_\ell (\gamma(n)) = 1 \right. \right\}$$

is a convex polyhedron, which has $\Omega$ vertices denoted by $\mathcal{H}^{(v)}$, $v = 1, 2, \ldots, \Omega$ in Equation (29) respectively.

From Equation (39), it’s clear that inequality constraint (38) is a convex constraint related to matrix $H$. Consequently it can only be verified at the $\Omega$ vertices of $\mathbb{H}$, producing inequality constraints (22).

The auxiliary inequality constraint (24) (i.e., $\Pi_{22} \leq 0$) can make inequality constraint (37) be a convex constraint that is related to $\Delta$. Consequently, when $\mathbb{D}$ is a convex polyhedron, under condition (26), inequality (37) can only be verified at $2^{m+1}$ vertices of $\mathbb{D}$, which produces inequality constraints (23).

Similarly, by changing $\Theta$ to $-\Theta$ in inequality constraint (22), inequality (33) can be attained. Therefore, we obtain inequality (25).

Theorem 3 works even if $\gamma(n) = 0$. When $\gamma(n) = 0$, the left side of Equation (25) becomes

$$\sum_{\ell = 1}^{\Omega} h_\ell (\gamma(n)) q_0^{\ell, 0}.$$  

This results in $q_0^{\ell, 0} > 0$ or $q_0^{\ell, 0} < 0$, $\ell = 1, 2, \ldots, \Omega$. Since $h_\ell (\gamma(n)) \geq 0$ and $\sum_{\ell = 1}^{\Omega} h_\ell (\gamma(n)) = 1$ (i.e., Equations (3) and (4)), it can be concluded that $\sum_{\ell = 1}^{\Omega} h_\ell (\gamma(n)) q_0^{\ell, 0} \neq 0$.

The computational complexity of Theorem 3 can be characterized by number of decision variables

$$\sum_{\ell = 1}^{\Omega} h_\ell (\gamma(n)) q_0^{\ell, 0}.$$

4. Numerical examples

We provide two numerical examples to illustrate the utility of our theoretical results. One is to show the method in Theorem 3, and the other is to design a feedback linearization controller by utilizing Theorem 1 to attain asymptotical output tracking with the disturbance rejection.

**Example 1.** Given a discrete-time quadratic TS fuzzy system (2) without disturbance (i.e., $d(n) = 0$) that has the following parameters: $m = 0$ and $p = 0$. Two fuzzy rules are used:

$$R^1 : \text{IF } y(n) \text{ is } P_0, \text{ THEN}$$

$$y(n + 1) = -5y(n) + (2 + a \, y(n)^2) \, u(n),$$

$$R^2 : \text{IF } y(n) \text{ is } N_0, \text{ THEN}$$

$$y(n + 1) = y(n) + (3 + b \, y(n)^2) \, u(n),$$

where $a$ and $b$ are undetermined constants and will be assigned later. The two sets are defined in Fig. 1 and Table 1. Product fuzzy AND operator is used for all the ANDs. The centroid defuzzifier is used (i.e., $\alpha = 1$).

**Solution:** We use Theorem 3 to check whether formula (25) holds for the specific system in Example 1.
We let scalars $\phi = -5$ and $\bar{\phi} = 5$. Then, inequality (26) becomes
\[ -5 \leq y(n) \leq 5 \]
and $\Delta$, $G_{11}$, $G_{21}$, and $G_{22}$ in Equations (27 and 28) are calculated as follows:
\[ \Delta = y(n), \quad G_{11} = 0, \quad G_{12} = 1, \]
\[ G_{21} = \begin{bmatrix} 0 \\ T \end{bmatrix}, \quad G_{22} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]
and the vertices $\Delta_i$ in inequality constraints (23) are
\[ \Delta^{(1)} = -5, \quad \Delta^{(2)} = 5. \]
Moreover, $\mathcal{H}(\nu), \nu = 1, 2$ and $\bar{\Theta}$ in Equations (29 and 30) are found to be
\[ \mathcal{H}^{(1)} = \begin{bmatrix} I_2 \\ 0_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{H}^{(2)} = \begin{bmatrix} 0_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \]
\[ \bar{\Theta}_1 = \begin{bmatrix} q_0^{1,0} \\ 0.5v_0^{1,0} \\ 0.5r_0^{1,0} \\ 1.0 \end{bmatrix} = \begin{bmatrix} 2 \\ a \end{bmatrix}, \]
\[ \bar{\Theta}_2 = \begin{bmatrix} q_0^{2,0} \\ 0.5v_0^{2,0} \\ 0.5r_0^{2,0} \\ 1.0 \end{bmatrix} = \begin{bmatrix} 3 \\ b \end{bmatrix}. \]
Take $a = 0$ and $b = 1$ as an example for making $\sum_{\ell=1}^{2} h_{\ell} (\bar{y}(n)) g_{\ell} (\bar{y}(n)) \neq 0$ clearly visible, which will be further explained later. By utilizing Theorem 3, we can find the solution $\Pi$ that satisfy $5$ (i.e., $\Omega + 2^m + 1 = 5$) inequality constraints (22–24) with $-\bar{\Theta}$ in (22) is
\[ \Pi = \begin{bmatrix} 1.4569 & 0 \\ 0 & -0.0244 \end{bmatrix}, \]
which means inequality (33) (i.e., $\sum_{\ell=1}^{2} h_{\ell} (\bar{y}(n)) g_{\ell} (\bar{y}(n)) > 0$) holds. In other words, formula (25) (i.e., $\sum_{\ell=1}^{2} h_{\ell} (\bar{y}(n)) g_{\ell} (\bar{y}(n)) \neq 0$) holds.

For the specific parameters, the computational complexity of this example is characterized by 3 decision variables, 7 LMIs. The CPU time for calculating the 7 LMIs is 0.0156 s.

**Example 2.** Assume that the following discrete-time quadratic TS fuzzy system (2) is to be controlled and there is a square disturbance signal (i.e., $d(n) \neq 0$): $m = 1$ and $p = 0$. Four fuzzy rules (i.e., $\Omega = 4$) are used with the $\ell$-th rule being
\[ R^{\ell}: \text{IF } y(n) \text{ is } M_{\ell}^{\ell} \text{ AND } y(n-1) \text{ is } M_{\ell}^{\ell}, \text{ THEN } y(n+1) = a_1^{\ell} y(n-1) + \left( q_0^{\ell} + v_0^{\ell} \right) y(n) + s_0^{\ell} y(n)^2 u(n) + d(n). \]

Actually, we can calculate that
\[ \sum_{\ell=1}^{2} h_{\ell} (\bar{y}(n)) g_{\ell} (\bar{y}(n)) = h_1 (\bar{y}(n)) (2 + 0) + h_2 (\bar{y}(n)) (3 + y(n)^2) = 2 (1 + h_2 (\bar{y}(n))) + h_2 (\bar{y}(n)) (3 + y(n)^2) = 2 + h_2 (\bar{y}(n)) (1 + y(n)^2) > 0. \]

From the above result, we can see that Theorem 3 is useful for determining which fuzzy system will not make the controller fail.

In the next example, we exhibit how to use our proposed method to solve the tracking control problem for the discrete-time quadratic TS fuzzy systems. The control objective is to find such an input sequence $\{u(n)\}$ that makes the output of the closed-loop system follow a user-desired step reference signal $y_r$ and attenuate a square disturbance signal at the same time.

<table>
<thead>
<tr>
<th>Table 2</th>
<th>The parameter values for the four fuzzy rules</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell$</td>
<td>$M_0^{\ell}/M_1^{\ell}$</td>
</tr>
<tr>
<td>1</td>
<td>$P_0/P_1$</td>
</tr>
<tr>
<td>2</td>
<td>$P_0/N_1$</td>
</tr>
<tr>
<td>3</td>
<td>$N_0/P_1$</td>
</tr>
<tr>
<td>4</td>
<td>$N_0/N_1$</td>
</tr>
</tbody>
</table>

The parameter values for the four fuzzy sets and four fuzzy rules are shown in Tables 2 and 3 and Fig. 1. Product fuzzy AND operator and the centroid defuzzifier are used.

**Solution:** First, we use Theorem 3 to check whether formula (25) holds for the fuzzy system. Let scalars $\phi$ and $\bar{\phi}$ in inequality (26) be the same as those in Example 1. $\Delta$, $G_{11}$, $G_{21}$, and $G_{22}$ in Equations (27 and 28) become

\[ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ a \end{bmatrix}, \quad \begin{bmatrix} 3 \\ b \end{bmatrix}. \]
The parameter values of the four membership functions (see Fig. 1)

<table>
<thead>
<tr>
<th>$i$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{P_1}$</td>
<td>-1</td>
<td>-1.1</td>
</tr>
<tr>
<td>$c_{P_2}$</td>
<td>0.8</td>
<td>1.7</td>
</tr>
<tr>
<td>$K_{P_1}$</td>
<td>0.5556</td>
<td>0.3571</td>
</tr>
<tr>
<td>$d_{P_1}$</td>
<td>0.5556</td>
<td>0.3929</td>
</tr>
<tr>
<td>$c_{N_1}$</td>
<td>-0.7</td>
<td>-0.6</td>
</tr>
<tr>
<td>$c_{N_2}$</td>
<td>0.9</td>
<td>1.4</td>
</tr>
<tr>
<td>$K_{N_1}$</td>
<td>-0.6250</td>
<td>-0.5</td>
</tr>
<tr>
<td>$d_{N_2}$</td>
<td>0.5625</td>
<td>0.7</td>
</tr>
</tbody>
</table>

$$\Delta = [y(n) , y(n - 1)]^T,$$

$$G_{11} = [0, 0] , \ G_{12} = 1 , \ G_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} , \ G_{22} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ,$$

and the vertices $\Delta^{(\sigma)} , \sigma = 1, 2, 3, 4$, in inequality constraints (23) are

$$\Delta^{(1)} = \begin{bmatrix} -5 \\ -5 \end{bmatrix} , \Delta^{(2)} = \begin{bmatrix} -5 \\ 5 \end{bmatrix} ,$$

$$\Delta^{(3)} = \begin{bmatrix} 5 \\ -5 \end{bmatrix} , \Delta^{(4)} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} .$$

Moreover, $\mathcal{H}^{(\nu)} , \nu = 1, 2, 3, 4$ and $\bar{\Theta}$ in Equations (29 and 30) are

$$\mathcal{H}^{(1)} = \begin{bmatrix} I_3 \\ 0_3 \end{bmatrix} , \ \mathcal{H}^{(2)} = \begin{bmatrix} 0_3 \\ I_3 \end{bmatrix} , \ \mathcal{H}^{(3)} = \begin{bmatrix} 0_3 \\ I_3 \end{bmatrix} , \ \mathcal{H}^{(4)} = \begin{bmatrix} I_3 \\ 0_3 \end{bmatrix} ,$$

$$\bar{\Theta} = \begin{bmatrix} q_0^1 & 0.5v_0^1 & 0.5v_1^1 \\ 0.5v_0^1 & 0 & 0.5v_1^1 \\ 0.5v_0^1 & 0.5v_1^1 & s_{1,1}^1 \end{bmatrix} = \begin{bmatrix} 1 & -0.5 & 0 \\ -0.5 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$

$$\bar{\Theta}_2 = \begin{bmatrix} q_0^2 & 0.5v_0^2 & 0.5v_1^2 \\ 0.5v_0^2 & 0 & 0.5v_1^2 \\ 0.5v_0^2 & 0.5v_1^2 & s_{1,1}^2 \end{bmatrix} = \begin{bmatrix} 3 & 1.5 & 0 \\ 1.5 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$

$$\bar{\Theta}_3 = \begin{bmatrix} q_0^3 & 0.5v_0^3 & 0.5v_1^3 \\ 0.5v_0^3 & 0 & 0.5v_1^3 \\ 0.5v_0^3 & 0.5v_1^3 & s_{1,1}^3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$

$$\bar{\Theta}_4 = \begin{bmatrix} q_0^4 & 0.5v_0^4 & 0.5v_1^4 \\ 0.5v_0^4 & 0 & 0.5v_1^4 \\ 0.5v_0^4 & 0.5v_1^4 & s_{1,1}^4 \end{bmatrix} = \begin{bmatrix} 2 & 0.5 & 0 \\ 0.5 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

By utilizing Theorem 3, we can find the solution $\Pi$ meeting 9 (i.e., $\Omega + 2^{-m+1} + 1 = 9$) inequality constraints (22–24) with $-\Theta$ in (22):

$$\Pi = \begin{bmatrix} 0.7435 & -0.0021 & 0 \\ -0.0021 & -0.0098 & 0 \\ 0 & 0 & -0.0147 \end{bmatrix} ,$$

which means inequality (33) (i.e., $\sum_{\ell = 1}^{4} h_{\ell}(\tilde{y}(n)) g_{0,\ell}(\tilde{y}(n)) > 0$) holds. Consequently, formula (25) (i.e., $\sum_{\ell = 1}^{4} h_{\ell}(\tilde{y}(n)) g_{0,\ell}(\tilde{y}(n)) \neq 0$) holds.

The computational complexity of this example is reflected by 6 decision variables and 9 LMI's that consume a total CPU time of 0.0312 s.

In the second part of the example, we utilize Theorem 1 to design a feedback linearization controller.

Rewriting the discrete-time quadratic TS fuzzy system in (40) to the state space form (10) produces

$$\bar{x}(n + 1) = A \bar{x}(n) + B \nu(n) + B d(n) ,$$

$$z(n) = C \bar{x}(n) ,$$

where $\bar{x}(n) = [x_1(n) , x_2(n)]^T = [y(n - 1) , y(n)]^T$, and

$$A = \begin{bmatrix} 0 & 1 \\ \hat{a}_1 & \hat{a}_0 \end{bmatrix} , \ B = [0, 1]^T , \ C = [0, 1]$$

Calculating using Equation (8), we obtain

$$\hat{a}_0 = 0, \ \hat{a}_1 = 4.3533 . \ \ \ \ \ (41)$$
We calculate the rank of the matrices \([B, AB]\) and 
\[
\begin{bmatrix}
I - A & B \\
-C & 0
\end{bmatrix}
\]
\[
\text{rank}[B, AB] = 2,
\]
\[
\begin{bmatrix}
I - AB & B
\end{bmatrix}
\]
\[
\text{rank}
\begin{bmatrix}
I - AB & C_0
\end{bmatrix}
\] = 3.

According to Theorem 2, we know that the augmented system 
\[
\begin{bmatrix}
\ddot{x}(n + 1) \\
\ddot{\xi}(n + 1)
\end{bmatrix}
= 
\begin{bmatrix}
0 & 1 & 0 \\
4.3533 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(n) \\
x_2(n) \\
\xi(n)
\end{bmatrix}
+ 
\begin{bmatrix}
1 \\
0
\end{bmatrix}
v(n)
- 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
y_r + 
\begin{bmatrix}
1 \\
0
\end{bmatrix}d(n),
\]
\[
z(n) = 
\begin{bmatrix}
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(n) \\
x_2(n) \\
\xi(n)
\end{bmatrix}
\]
is completely controlled and the asymptotic tracking controller can be designed for it.

Placing the desired closed-loop poles at 
\[
x_1 = 0.6, \ x_2 = 0.2, \ x_3 = 0.8
\]
and using the pole placement design method, we obtain the following controller gain:
\[
\bar{K} = [\bar{K}_x, \ K_q] = [K_1, K_0, K_q]
= [-4.2573, -1, -0.5760].
\]

As a result, controller (16), which is in a compact form for system (40), becomes 
\[
u(n) = 
\sum_{i=0}^{1} K_i y(n-i) + K_q \xi(n) + \sum_{i=0}^{1} \hat{a}_i y(n-i)
\sum_{\ell=1}^{4} h_\ell(\bar{y}(n))(q_{\ell,0}^0 + v_{\ell,0}^0 y(n) + \ell v_{\ell,0}^0 y(n)^2)
\sum_{\ell=1}^{4} h_\ell(\bar{y}(n))(a_{\ell,1}^0 y(n-1))
\sum_{\ell=1}^{4} h_\ell(\bar{y}(n))(q_{\ell,0}^0 + v_{\ell,0}^0 y(n) + \ell v_{\ell,0}^0 y(n)^2)
\]

Both the numerator and denominator can be easily derived based on Equations (41 and 42), and Tables 2 and 3.
Figures 2 to 4 show the control performances. The initial states are selected to be $y(n) = 0.2$, $y(n - 1) = 0.1$ and the amplitude of the square disturbance signal is 0.1 which starts at $n = 121$ and ends at $n = 145$. The system responses shown in the figures confirm that feedback linearization controller (43) can indeed make system output asymptotically track step reference signal and restrain the effect of the square disturbance signal on system output effectively.

5. Conclusion

We present an asymptotical output tracking control design method for a general class of the discrete-time TS fuzzy systems with quadratic rule consequents in the presence of square disturbance signal. The method utilizes the feedback linearization method and is combined with a PI controller. The design is analytical. Furthermore, we develop a sufficient condition by utilizing the full block S-procedure method to check whether any given quadratic TS fuzzy system will cause a tracking controller to fail to work due to its instability, which is an important issue that has never been tackled in the literature.

One limitation of our current controller design framework is that it covers step reference signal and square disturbance signal. Effort is needed to extend the design approach to cover other reference and disturbance signals.

Acknowledgments

This work was supported in part by the National Natural Science Foundation of China under grant number 61304006 and 61273095.

References


