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Analytical Structure of a Fuzzy Controller with Linear Control Rules

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ABSTRACT

We prove that a fuzzy controller with linear control rules and Mamdani minimum inference is the sum of a global two-dimensional multilevel relay and a local nonlinear proportional-integral (PI) controller. The properties of the nonlinear gains of the local PI controller are investigated. Moreover, it is proven that, as the number of input fuzzy sets approaches ∞, the global multilevel relay approaches a global linear PI controller while the local nonlinear PI controller disappears.

INTRODUCTION

Fuzzy controllers are linguistic rule-based controllers that can effectively be implemented to solve complex control problems when precise process mathematical models are unavailable. There have been many successful fuzzy control applications, especially in Japan, ranging from industrial applications to consumer products [14]. In contrast to application, however, there is only a little solid fuzzy control theory. For a comprehensive review, the reader is referred to a recent survey paper by Lee [9]. A new trend in fuzzy control research is to combine fuzzy controllers with neural networks so that neural networks can learn membership functions and fuzzy control rules for fuzzy controllers (e.g., [6, 13]). Summarily speaking, compared to the well-developed classic control theory, fuzzy control theory lacks precise analysis methods (including stability analysis) and systematic design procedures.

In order to analyze and design fuzzy controllers, analytical structure of fuzzy controllers in relation to classic controllers is essential (e.g., [5, 8, 10, 12, 15, 17, 19, 20]). Fuzzy controllers with linear control rules have been studied extensively due to their importance [1–4, 18]. Recently, we proved that a fuzzy controller with linear control rules and Larsen or drastic product inference was the sum of a global two-dimensional multilevel relay

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and a local nonlinear PI controller [18]. Such an expression of the analytical structure of the fuzzy controller is technically useful and desirable because it closely relates the fuzzy controller to classic controllers (multilevel relay and PI controller). As a result, analysis and design methods in classic control theory may be applied to the fuzzy controller. Moreover, such a representation has an interesting property: As the number of input fuzzy set approaches ∞, the global multilevel relay approaches a linear PI controller while the local nonlinear PI controller disappears [18].

In this paper, we will extend our effort to analytically derive the structure of the fuzzy controller that uses linear control rules and Mamdani minimum inference. We will also investigate properties of the resultant structure.

2. ANALYTICAL ANALYSIS OF THE STRUCTURE OF A FUZZY CONTROLLER WITH LINEAR CONTROL RULES

2.1. COMPONENTS OF THE FUZZY CONTROLLER

Inputs of the fuzzy controller are error and rate change of error (rate, for short) of process output. The scaled inputs are

$$e^* = \mathrm{GE}\cdot e(nT) = \mathrm{GE}(\text{setpoint} - y(nT)),$$

$$r^* = \mathrm{GR}\cdot r(nT) = \mathrm{GR}(e(nT) - e(nT-T)),$$

where GE and GR are scalars for the error and rate, respectively, $y(nT)$ is process output (nT is sampling time with n being a positive integer and T being a sampling period), $e(nT-T)$ is error at sampling time $(n-1)T$, and the setpoint is the desired process output.

The scaled error and rate are fuzzified, respectively, by input fuzzy sets. The input fuzzy sets are triangular-shaped. There are J $(J>0)$ fuzzy sets for positive e^* and r^*, J fuzzy sets for negative e^* and r^*, and one fuzzy set for near zero e^* and r^*. Figure 1 illustrates the definitions of the input fuzzy sets. There are total

$$N = 2J + 1$$

fuzzy sets for e^* and r^*. A fuzzy set for e^* (or r^*) is denoted as E_i (or R_j) and the corresponding membership is designated as $\mu_i(e^*)$ [or $\mu_j(r^*)$]. It

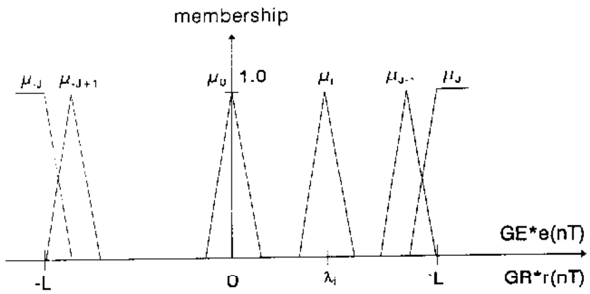

Fig. 1. Illustrative definitions of the triangular-shaped input fuzzy sets. There are total $N=2J+1$ input fuzzy sets. The space between the central values (λ_i) of two adjacent fuzzy sets is S. It should be noted that the fuzzy sets for e^* and the fuzzy sets for r^* are identical.

should be noted that

$$\mu_i(e^*)+\mu_{i+1}(e^*)=1 \quad \text{and} \quad \mu_j(r^*)+\mu_{j+1}(r^*)=1.$$

Denote the central value of μ_i as λ_i and define $\lambda_{-J}=-L$, $\lambda_0=0$, and $\lambda_J=L$ (the central value of μ_i is the value of e^* at which μ_i is maximum, which is 1 in this paper). Let the space between the central values of two adjacent fuzzy sets be equal. Then the space is

$$S=\frac{L}{J}$$

and consequently the central value of μ_i is $\lambda_i=i\cdot S$. Note that the central value of μ_j is $\lambda_j=j\cdot S$.

There are $4J+1$ $(2N-1)$ output fuzzy sets for incremental output of the fuzzy controller, Δu. There are $2J$ fuzzy sets for positive Δu, $2J$ fuzzy sets for negative Δu, and one fuzzy set for near zero Δu. The shapes of the membership functions of the fuzzy sets are identical. An output fuzzy set is denoted as U_k whose membership function $\mu_k(\Delta u)$ is trapezoidal-shaped and symmetrical about its central value γ_k, as shown in Fig. 2. We let $\gamma_{-2J}=-H$, $\gamma_0=0$, and $\gamma_{2J}=H$. The space between the central values of two adjacent output fuzzy sets is equal, which is

$$V=\frac{H}{2J}=\frac{H}{N-1}.$$

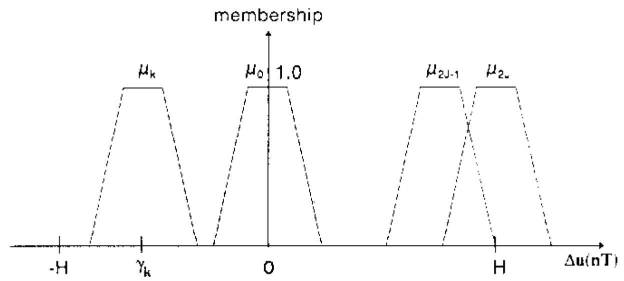

Fig. 2. Illustrative definitions of the trapezoidal-shaped output fuzzy sets. There are total $4J+1$ (or $2N-1$) output fuzzy sets. The space between the central values (γ_k) of two adjacent output fuzzy sets is V. Note that $2A$ and $2V$ are the upper side and lower side of each output fuzzy set, respectively. $\theta = A/V$ as defined in (2.1).

Hence, the central value of $\mu_k(\Delta u)$ is $\gamma_k = k \cdot V$. To define the shape of the trapezoid, a parameter

$$\theta = \frac{A}{V} \tag{2.1}$$

is employed, which is constrained by $\theta \leq 0.5$ to avoid overlay between upper sides of two adjacent output fuzzy sets. Note that $2A$ and $2V$ are the upper side and lower side of each output fuzzy set, respectively.

N^2 control rules are used to cover $N \times N$ possible combinations of the input fuzzy sets. The control rules in this study comply with the following law:

$$\text{IF } e^* \text{ is } E_i \text{ and } r^* \text{ is } R_j \text{ THEN } \Delta u \text{ is } U_{i+j}. \tag{2.2}$$

The control rule is called a linear control rule because the linear function is employed to relate the indexes of the input fuzzy sets to the index of the output fuzzy set.

If two control rules generate two different memberships, say μ_1 and μ_2, for a same output fuzzy set, Lukasiewicz fuzzy logic OR,

$$\mu = \text{Min}(\mu_1 + \mu_2, 1), \tag{2.3}$$

is used to get combined membership for the output fuzzy set, because the conditions being ORed are maximally negatively correlated [16].

The widely used Mamdani minimum inference, whose definition is illustrated in Fig. 3, is employed to reason the output fuzzy sets from the input fuzzy sets in the control rules. In Fig. 3, $\mu(i,j)$ is the membership for U_{i+j}, which is obtained from E_i and R_j according to the control rule, using Zadeh fuzzy logic AND (Min operation). The shaded area is

$$S(\mu(i,j)) = (2 - \mu(i,j) + \theta \cdot \mu(i,j)) \cdot \mu(i,j) \cdot V. \tag{2.4}$$

The popular center of gravity defuzzification algorithm is used for defuzzifying the output fuzzy sets generated by the control rules. Because the shapes of the membership functions of the output fuzzy sets are the same, the global centroid can be calculated from the local centroids which are always the central values of the output fuzzy sets involved. Hence, the scaled crisp incremental output (the global centroid) $\mathrm{GU} \cdot \Delta u(nT)$ is calculated as

$$\mathrm{GU} \cdot \Delta u(nT) = \mathrm{GU} \frac{\sum_{\mu(i,j) \neq 0} S(\mu(i,j)) \cdot \gamma_{i+j}}{\sum_{\mu(i,j) \neq 0} S(\mu(i,j))}, \tag{2.5}$$

where GU is an output scalar. The new output of the controller is

$$u(nT) = u(nT - T) + GU \cdot \Delta u(uT).$$

2.2. *MAIN RESULTS*

THEOREM 1. *The structure of the fuzzy controller using the linear control rules and Mamdani minimum inference is the sum of a global two-dimensional multilevel relay and a local nonlinear* PI *controller.*

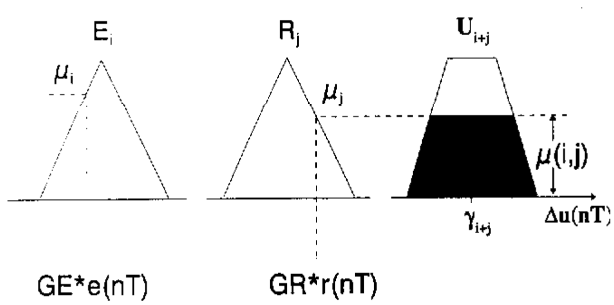

Fig. 3. Illustrative definition of the widely used Mamdani minimum inference. The trapezoidal-shaped output fuzzy sets are defined in Fig. 2. $\mu(i,j)$ is the membership for U_{i+j}, which is obtained from E_i and R_j according to the linear control rule (2.2) using Zadeh fuzzy logic AND. The shaded area can be calculated based on the formula (2.4).

Proof. We will prove the theorem when (1) both e^* and r^* are within the interval $[-L, L]$ and (2) either e^* or r^* is outside the interval $[-L, L]$.

(1) Both e^* and r^* are within the interval $[-L, L]$: At any sampling time nT, the scaled inputs must satisfy

$$iS \le e^* \le (i+1)S \quad \text{and} \quad jS \le r^* \le (j+1)S.$$

After fuzzification, the memberships for the members E_i, E_{i+1}, R_j and R_{j+1} are obtained:

$$\mu_i(e^*) = -\frac{e^* - (i+1)S}{S}, \qquad \mu_{i+1}(e^*) = \frac{e^* - i \cdot S}{S},$$

$$\mu_j(r^*) = -\frac{r^* - (j+1)S}{S}, \qquad \mu_{j+1}(r^*) = \frac{r^* - j \cdot S}{S}.$$

Membership for all the other input fuzzy sets is zero. Consequently, only the following four linear control rules are activated:

IF e^* is E_{i+1} and r^* is R_{j+1} THEN Δu is U_{i+j+2}, $\quad(\mathrm{r}_1)$

IF e^* is E_{i+1} and r^* is R_j THEN Δu is U_{i+j+1}, $\quad(\mathrm{r}_2)$

IF e^* is E_i and r^* is R_{j+1} THEN Δu is U_{i+j+1}, $\quad(\mathrm{r}_3)$

IF e^* is E_i and r^* is R_j THEN Δu is U_{i+j}. $\quad(\mathrm{r}_4)$

To determine the results of Zadeh fuzzy logic AND used in Mamdani minimum inference, a square is configured by the intervals $[iS, (i+1)S]$ and $[jS, (j+1)S]$ as shown in Fig. 4. The square is divided into four input combination (IC) regions. In each region, the Min operation is carried out for the control rules r_1 to r_4 and the outcomes are illustrated in the first four rows of Table 1. Because the control rules r_2 and r_3 generate two memberships for the same member, U_{i+j+1}, Lukasiewicz fuzzy logic OR (2.3) is used to calculate the combined membership for U_{i+j+1}. For the IC1 to IC4 regions, it can be proven easily that

$$\mu_i(e^*) + \mu_j(r^*) \le 1 \quad \text{and} \quad \mu_{i+1}(e^*) + \mu_{j+1}(r^*) \le 1.$$

Therefore, the combined membership for U_{i+j+1} is always the sum of the membership being ORed. Substitute the first four rows of the memberships in Table 1 into the defuzzification algorithm (2.5) and simplify the

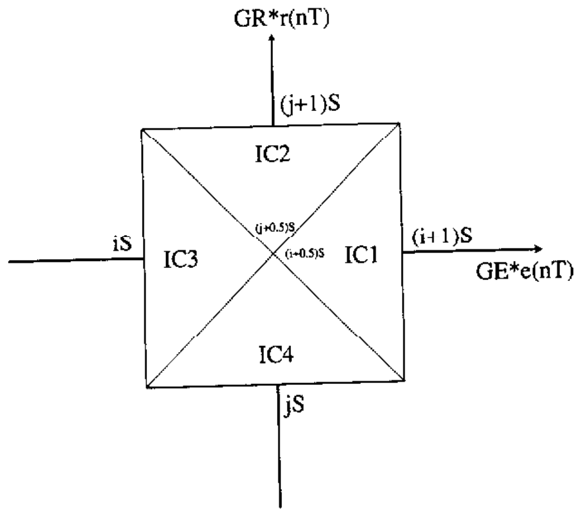

Fig. 4. Possible input combinations (IC) of scaled error e^* and scaled rate change of error r^* of process output that must be considered to carry out Zadeh fuzzy logic AND in the control rules r_1 to r_4 when both e^* and r^* are within the interval $[-L, L]$.

resulting expressions. The scaled incremental output $\mathrm{GU}\cdot\Delta u(nT)$ is found as follows:

$$\begin{aligned}
\mathrm{GU}\cdot\Delta u(nT) &= (i+j+1)\frac{\mathrm{GU}\cdot H}{N-1} + \beta_1 \\
&\quad \times [(\mathrm{GE}\cdot e(nT) - (i+0.5)S) + (\mathrm{GR}\cdot r(nT) - (j+0.5)S)] \\
&\quad \times \frac{\mathrm{GU}\cdot H}{2L} \\
&= (i+j+1)\frac{\mathrm{GU}\cdot H}{N-1} \\
&\quad + \left[K_i\left(e(nT) - \frac{(i+0.5)S}{\mathrm{GE}}\right) + K_p\left(r(nT) - \frac{(j+0.5)S}{\mathrm{GR}}\right)\right],
\end{aligned}$$

TABLE 1

Results of Evaluating Zadeh Fuzzy Logic AND in the Control Rules r_1 to r_4 for all 12 Combinations of the Scaled Inputs

Input combinations (IC) of the scaled error and rate	Memberships of the output fuzzy sets obtained by evaluating the input fuzzy set in the control rules r_1 to r_4 using Zadeh fuzzy logic AND			
	r_1	r_2	r_3	r_4
IC1	$\mu_{j+1}(r^*)$	$\mu_j(r^*)$	$\mu_i(e^*)$	$\mu_i(e^*)$
IC2	$\mu_{i+1}(e^*)$	$\mu_j(r^*)$	$\mu_i(e^*)$	$\mu_j(r^*)$
IC3	$\mu_{i+1}(e^*)$	$\mu_{i+1}(e^*)$	$\mu_{j+1}(r^*)$	$\mu_j(r^*)$
IC4	$\mu_{j+1}(r^*)$	$\mu_{i+1}(e^*)$	$\mu_{j+1}(r^*)$	$\mu_i(e^*)$
IC5	$\mu_{j+1}(r^*)$	$\mu_j(r^*)$	0	0
IC6	$\mu_{i+1}(e^*)$	0	$\mu_i(e^*)$	0
IC7	0	0	$\mu_{j+1}(r^*)$	$\mu_j(r^*)$
IC8	0	$\mu_{i+1}(e^*)$	0	$\mu_i(e^*)$
IC9	1	0	0	0
IC10	0	0	1	0
IC11	0	0	0	1
IC12	0	1	0	0

[a]The first four rows are the evaluation results when the scaled error and rate change of error are within the interval $[-L, L]$. The remaining eight rows are the evaluation results when the scaled error or rate change of error of process output is outside $[-L, L]$. The input combinations of the scaled inputs are shown graphically in Figs. 4 and 5.

where

$$\beta_1 = \frac{\begin{array}{l}(1+\theta)S^2+(1-\theta)S \\ \qquad \times\left|(\mathrm{GE}\cdot e(nT)-(i+0.5)S)-(\mathrm{GR}\cdot r(nT)-(j+0.5)S)\right|\end{array}}{\begin{array}{l}(3+\theta)S^2-2(1+\theta)S\times \text{input} \\ \quad -2(1-\theta)\left[(\mathrm{GE}\cdot e(nT)-(i+0.5)S)^2+(\mathrm{GR}\cdot r(nT)-(j+0.5)S)^2\right]\end{array}},$$

$$\text{input} = \begin{cases} |\mathrm{GE}\cdot e(nT)-(i+0.5)S|, & \text{IC1 and IC3}, \\ |\mathrm{GR}\cdot r(nT)-(j+0.5)S|, & \text{IC2 and IC4}, \end{cases} \tag{2.6}$$

$$K_p = \frac{\beta_1\cdot \mathrm{GR}\cdot \mathrm{GU}\cdot H}{2L}, \tag{2.7}$$

$$K_i = \frac{\beta_1\cdot \mathrm{GE}\cdot \mathrm{GU}\cdot H}{2L}. \tag{2.8}$$

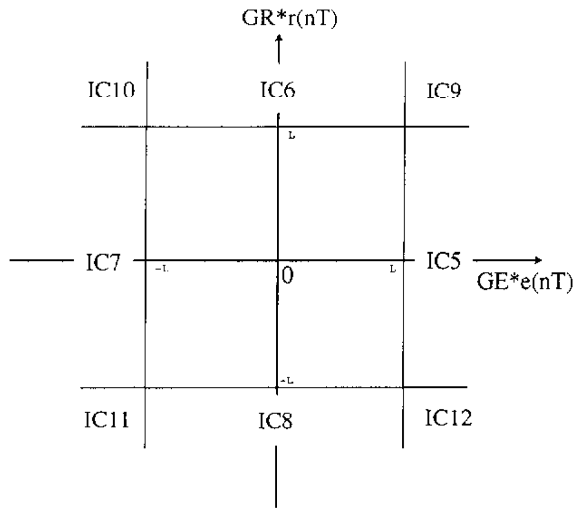

Fig. 5. Possible input combinations (IC) of scaled error e^* and scaled rate change of error r^* of process output that must be considered to carry out Zadeh fuzzy logic AND in the control rules r_1 to r_4 when either e^* or r^* is outside the interval $[-L, L]$.

$GU \cdot \Delta u(nT)$ consists of two parts. The first part is $(i+j+1)GU \cdot H/(N-1)$, which is a global two-dimensional multilevel relay with respect to the inputs i and j. The relay is called a global relay because its action is calculated according to i and j with respect to the origin of the scaled input state space $(0, 0)$.

The second part of $GU \cdot \Delta u(nT)$ is a local nonlinear PI controller with the nonlinear proportional gain (2.7) and integral gain (2.8). The nonlinear PI controller is named a local controller because its action is càlculated according to the relative position of the current scaled input state $(GE \cdot e(nT), GR \cdot r(nT))$ with respect to the center of the square $((i+0.5)S, (j+0.5)S)$ in which the current scaled input state lies. It should be noted that the nonlinear PI controller has a local and changing steady state $((i+0.5)S/GE, (j+0.5)S/GR)$.

(2) Either e^* or r^* is outside the interval $[-L, L]$: The scaled input state space outside the square configured by $[-L, L]$ on the scaled error axis and $[-L, L]$ on the scaled rate axis is divided into eight IC regions, as shown in Fig. 5. In different IC regions, the outcomes of Zadeh fuzzy logic AND in Mamdani minimum inference are found as shown from the fifth to the last row in Table 1. Using the same method described above,

$\mathrm{GU}\cdot\Delta u(nT)$ can analytically be derived as

$$\mathrm{GU}\cdot\Delta u(nT)=\begin{cases}(J+j)\dfrac{\mathrm{GU}\cdot H}{N-1}+\beta_2(\mathrm{GR}\cdot r(nT)-jS)\dfrac{\mathrm{GU}\cdot H}{2L}\\ \quad=(J+j)\dfrac{\mathrm{GU}\cdot H}{N-1}+K_p\Big(r(nT)-\dfrac{jS}{\mathrm{GR}}\Big), & \mathrm{IC5},\\ (i+J)\dfrac{\mathrm{GU}\cdot H}{N-1}+\beta_2(\mathrm{GE}\cdot e(nT)-iS)\dfrac{\mathrm{GU}\cdot H}{2L}\\ \quad=(i+J)\dfrac{\mathrm{GU}\cdot H}{N-1}+K_i\Big(e(nT)-\dfrac{iS}{\mathrm{GE}}\Big), & \mathrm{IC6},\\ (-J+j)\dfrac{\mathrm{GU}\cdot H}{N-1}+\beta_2(\mathrm{GR}\cdot r(nT)-jS)\dfrac{\mathrm{GU}\cdot H}{2L}\\ \quad=(-J+j)\dfrac{\mathrm{GU}\cdot H}{N-1}+K_p\Big(r(nT)-\dfrac{jS}{\mathrm{GR}}\Big), & \mathrm{IC7},\\ (i-J)\dfrac{\mathrm{GU}\cdot H}{N-1}+\beta_2(\mathrm{GE}\cdot e(nT)-iS)\dfrac{\mathrm{GU}\cdot H}{2L}\\ \quad=(i-J)\dfrac{\mathrm{GU}\cdot H}{N-1}+K_i\Big(e(nT)-\dfrac{iS}{\mathrm{GE}}\Big), & \mathrm{IC8},\end{cases}\tag{2.9}$$

where

$$\beta_2=\frac{(3+\theta)S^2-2(1-\theta)S\cdot\mathrm{input}}{(3+\theta)S^2-4(1-\theta)\cdot(\mathrm{input})^2},$$

$$\mathrm{input}=\begin{cases}\mathrm{GE}\cdot e(nT)-(i+0.5)S, & \mathrm{IC6\ and\ IC8},\\ \mathrm{GR}\cdot r(nT)-(j+0.5)S, & \mathrm{IC5\ and\ IC7}\end{cases},\tag{2.10}$$

$$K_p=\frac{\beta_2\cdot\mathrm{GR}\cdot\mathrm{GU}\cdot H}{2L}\quad\text{and}\quad K_i=\frac{\beta_2\cdot\mathrm{GE}\cdot\mathrm{GU}\cdot H}{2L}.$$

The fuzzy controller becomes the sum of a global one-dimensional multilevel relay with respect to j and a local nonlinear proportional (P) controller with a local and changing steady state (jS/GR) when the scaled inputs are in the IC5 and IC7 regions. When the scaled inputs are in the IC6 and IC8 regions, the fuzzy controller is the sum of a global one-dimensional multilevel relay with respect to i and a local nonlinear integral (I) controller with a local and changing steady state (iS/GE).

The fuzzy controller generates its maximum increment ($\mathrm{GU} \cdot \Delta u(nT) = \mathrm{GU} \cdot \mathrm{H}$) and maximum decrement ($\mathrm{GU} \cdot \Delta u(nT) = -\mathrm{GU} \cdot H$) in the IC11 and IC9 regions, respectively. For the IC10 and IC12 regions, the increment is zero ($\mathrm{GU} \cdot \Delta u(nT) = 0$). ■

Theorem 1 can be interpreted geometrically. A fuzzy control rule defines a fuzzy patch in state space [7]. When the linear fuzzy control rules are used, fuzzy patches cover a plane in three-dimensional state space spanned by e^*, r^*, and $\mathrm{GU} \cdot \Delta u(nT)$. The output of the global two-dimensional multilevel relay is the centers of the patches in the direction of $\mathrm{GU} \cdot \Delta u(nT)$. Each patch is a local nonlinear PI controller that superimposes its local control action on the output of the global multilevel relay (the center of the patch). The local control action is calculated according to the current position of the state, (e^*, r^*), with respect to the center of the patch on $e^* - r^*$ plane, $((i+0.5)S, (j+0.5)S)$.

In practice, e^* and r^* are usually managed to stay within $[-L, L]$ in order to take full advantage of nonlinearity and dynamics of the fuzzy controller. Therefore, in the following analysis, properties of the fuzzy controller when both e^* and r^* are inside $[-L, L]$ will be investigated. Properties of the fuzzy controller when either e^* or r^* is outside $[-L, L]$ can be analyzed similarly, but will not be attempted for brevity.

As the structure of the fuzzy controller shows, the proportional gain K_p and integral gain K_i of the local nonlinear PI controller change with fuzzy controller inputs. They have the following properties.

THEOREM 2. *When both e^* and r^* are inside the interval $[-L, L]$,*

$$\frac{(1+\theta)\mathrm{GR} \cdot \mathrm{GU} \cdot H}{2(3+\theta)L} \leq K_p \leq \frac{\mathrm{GR} \cdot \mathrm{GU} \cdot H}{(1+\theta)L}, \tag{2.11}$$

$$\frac{(1+\theta)\mathrm{GE} \cdot \mathrm{GU} \cdot H}{2(3+\theta)L} \leq K_i \leq \frac{\mathrm{GE} \cdot \mathrm{GU} \cdot H}{(1+\theta)L} \tag{2.12}$$

For a given θ, the ratio of maximizing $K_p(K_i)$ to minimum $K_p(K_i)$ is

$$\varphi = \frac{K_p^{\max}}{K_p^{\min}} = \frac{K_i^{\max}}{K_i^{\min}} = \frac{2(3+\theta)}{(1+\theta)^2}. \tag{2.13}$$

For $0 \leq \theta \leq 0.5$,

$$3.11 \leq \varphi \leq 6. \tag{2.14}$$

Proof. β_1 reaches its minimum

$$\beta_1^{\min} = \frac{1+\theta}{3+\theta}$$

when $\mathrm{GE}\cdot e(nT) = (i+0.5)S$ and $\mathrm{GR}\cdot r(nT) = (j+0.5)S$. β_1 achieves its maximum

$$\beta_1^{\max} = \frac{2}{1+\theta}$$

either when $\mathrm{GE}\cdot e(nT) = (i+0.5)S$ and $\mathrm{GR}\cdot r(nT) = -(j+0.5)S$ or when $\mathrm{GE}\cdot e(nT) = -(i+0.5)S$ and $\mathrm{GR}\cdot r(nT) = (j+0.5)S$. Therefore,

$$\frac{1+\theta}{3+\theta} \le \beta_1 \frac{2}{1+\theta}. \tag{2.15}$$

Because of (2.7) and (2.8), inequalities (2.11) and (2.12) are true. Also, note that

$$\frac{K_p^{\max}}{K_p^{\min}} = \frac{K_i^{\max}}{K_i^{\min}} = \frac{\beta_1^{\max}}{\beta_1^{\min}}.$$

The proof of (2.13) and (2.14) follows. ■

It should be pointed out that when either e^* or r^* is outside $[-L, L]$, the proportional gain K_p and integral gain K_i of the local nonlinear P or I controller also change with fuzzy controller inputs and have similar properties.

As mentioned above, the maximum increment of the fuzzy controller is $\mathrm{GU}\cdot H$ and maximum decrement is $-\mathrm{GU}\cdot H$. Hence, the absolute value of maximum change is $\mathrm{GU}\cdot H$. The absolute value of the maximum control action from the global relay is

$$\frac{N-2}{N-1}\mathrm{GU}\cdot H, \tag{2.16}$$

which happens when $i=j=J-1$ or $i=j=-J$. Consequently, the absolute value of the maximum control action from the local nonlinear PI controller is

$$\mathrm{GU}\cdot H - \frac{N-2}{N-1}\mathrm{GU}\cdot H = \frac{\mathrm{GU}\cdot H}{N-1}. \tag{2.17}$$

It can be easily seen from equations (2.16) and 2.17) that the global relay plays a dominant role in the fuzzy control action while the role of the local PI controller is to locally fine adjust the control action generated by the global relay. The larger the N is, the more the global relay contributes and the less the local nonlinear PI controller contributes. When N approaches ∞, the structure of the fuzzy controller changes fundamentally, according to the following theorem.

THEOREM 3 (Limit theorem). *When $N \to \infty$, the local nonlinear* PI *controller disappears and the global two-dimensional multilevel relay becomes a global linear* PI *controller. That is,*

$$\mathrm{GU} \cdot \Delta u(nT) = K_i \cdot e(nT) + K_p \cdot r(nT), \tag{2.18}$$

where

$$K_p = \frac{\mathrm{GR} \cdot \mathrm{GU} \cdot H}{2L}, \tag{2.19}$$

$$K_i = \frac{\mathrm{GE} \cdot \mathrm{GU} \cdot H}{2L}. \tag{2.20}$$

Proof. According to (2.15), β_1 is a finite number and is independent of N. Therefore, the local nonlinear PI controller disappears when $N \to \infty$ because

$$\mathrm{GE} \cdot e(nT) - (i + 0.5)S \to 0 \quad \text{and} \quad \mathrm{GR} \cdot r(nT) - (j + 0.5)S \to 0.$$

See [18] for the proof that the global relay becomes the linear PI controller described in (2.18) with the proportional gain (2.19) and integral gain (2.20). ■

Theorem 3 can be explained in terms of fuzzy patches as well. The patches become smaller and smaller when the number of input fuzzy sets increases because the base of each input fuzzy set decreases. Note also that, as the patches become smaller and smaller, the distance between adjacent centers of the patches are shorter and shorter. As an extreme, when the number of input fuzzy sets approaches ∞, all the patches degenerate to points while all the centers of the patches connect each other, forming a plane that is the global linear PI controller (2.18).

It should be pointed out that the gobal-dimensional multilevel relays in (2.9) become a linear P controller for the IC5 and IC7 regions and a linear I controller for the IC6 and IC8 regions as $N \to \infty$.

3. DISCUSSION AND CONCLUSIONS

Although the global multilevel relay in this paper is identical to the global multilevel relay of the fuzzy controller using the linear control rules and Larsen product inference or drastic product inference [18], the expressions of the nonlinear proportional gain and integral gain were dramatically different because the expression of β_1 was dramatically different. The expression of β_1 in [18] was simply $\beta_1 = S/(S - \text{input})$, where the input is defined in (2.6). The range of β_1 was independent of θ and $1 \le \beta_1 \le 2$. Consequently, $\varphi \equiv 2$. A wider range of β_1 represents a wider range of change of K_p and K_i with the inputs. The ranges of change of K_p and K_i in this study ($3.11 \le \varphi \le 6$) are much larger than those in the previous study ($\varphi \equiv 2$).

Mizumoto studied 12 different inference methods for fuzzy control, using computer simulation and a first-order model with a time delay [11]. Lately, we theoretically proved that Mamdani minimum inference, Larsen product inference, and drastic inference were the only proper inferences for fuzzy control purpose among the 12 inference methods [20]. Based on the results in [18], [20], and this paper, we can now conclude that the structures of the fuzzy controllers using linear control rules are the sum of a global two-dimensional multilevel relay and a local nonlinear PI controller with different nonlinear proportional gain and integral gain corresponding to different inference methods. As the number of input fuzzy sets approaches ∞, the global multilevel relay approaches a regular linear PI controller while the control action from the local nonlinear PI controller approaches zero.

Finally, we like to point out that if the fuzzy controller in this paper uses $GU \cdot \Delta\, u(nT)$, instead of $U(nT) = U(nT - T) + GU \cdot \Delta u(nT)$, as its output at sampling time nT, its structure will be the sum of a global two-dimensional multilevel relay and local nonlinear proportional-derivative (PD) controller. This is also true for the fuzzy controllers in our previous papers [18, 20]. The fuzzy controllers in [20] will become nonlinear PD controllers with variable gains.

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