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# Typical Takagi–Sugeno PI and PD fuzzy controllers: analytical structures and stability analysis

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**Abstract**

In the first part of this paper, we investigate explicit structure of the typical Takagi–Sugeno (TS) fuzzy PI and PD controllers. By typical, we mean the use of at least three trapezoidal or triangular input fuzzy sets for each input variable, fuzzy rules with linear consequent, Zadeh fuzzy logic AND operator and the centroid defuzzifier. This configuration covers many TS fuzzy controllers in the literature. We mathematically prove these fuzzy controllers to be nonlinear PI (or PD) controllers with proportional-gain and integral-gain (or derivative-gain) changing with output of the controlled system. We reveal these fuzzy controllers to be inherently nonlinear gain scheduling PI (or PD) controllers with different variable gains in different regions of input space. The gains constantly change even in the same region, and switch continuously and smoothly between adjacent regions. The explicit structural expressions of the fuzzy controllers are derived and their characteristics, including the bounds and geometrical shape of the gain variation, are studied. In the second part of the paper, we apply the Small Gain Theorem to the derived structures of the fuzzy controllers and establish a sufficient global BIBO stability criterion for nonlinear systems controlled by the fuzzy controllers. Local stability is also examined.



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## 1. Introduction

Fuzzy control was based on the principles put forward by Zadeh [32,33]. There exist two major types of fuzzy controllers, namely Mamdani fuzzy controllers, introduced in 1974 [13], and Takagi–Sugeno (TS) fuzzy controllers, first developed in 1985 [17]. They mainly differ in the consequent of fuzzy rules: the former uses fuzzy sets whereas the latter employs (linear) functions. For the historical reason, Mamdani fuzzy controllers have received wider attention [21,24] and their analytical structures, including some rather complicated ones, have been revealed in relation to classical controllers through rigor derivation (e.g., [2–4,9,11,12,25,26,31]). Summarily, most Mamdani fuzzy controllers, if not all, are nonlinear controllers with variable gains. In particular, the Mamdani fuzzy PI, PD and PID controllers are, respectively, nonlinear PI, PD and PID controllers with variable gains [7,25,26,31]. Furthermore, analysis and design techniques for the Mamdani fuzzy control systems have been developed (e.g., [1,10,14,23]).

More and more interest appears to shift toward TS fuzzy controllers in the recent years, as evidenced by the increasing number of papers in this direction. Far fewer mathematical results exist for TS fuzzy controllers than do for Mamdani fuzzy controllers, notably those on TS fuzzy control system stability [19,20,22]. Analytical structure of TS fuzzy controllers was unknown prior to our recent work [6,28–30], in which we revealed explicitly the structures of some classes of TS fuzzy controllers with respect to the conventional controllers, including the PID controllers. We proved the general TS fuzzy controllers to be nonlinear controllers with variable gains [27]. This general class of fuzzy controllers used any type of continuous input fuzzy sets, any type of fuzzy logic AND operator, any fuzzy rules with linear rule consequent and the generalized defuzzifier containing the popular centroid defuzzifier as a special case. This configuration was general and many TS fuzzy controllers in the literature were its subset. Due to the generality in the configuration, the expressions of the general fuzzy controllers were mathematically not derivable. The difficulty lay in derivation of the analytical expressions of the variable gains, not the general structure of the fuzzy controllers which had been obtained already in [27]. To make the derivation mathematically feasible, the components of the fuzzy controllers must be specific (e.g., triangular input fuzzy sets, Zadeh fuzzy logic AND operator, etc.) even then, mathematical deviation is not guaranteed to be doable. On the other hand, when the derivation is doable, it is a technically challenging process.

To fulfill the structure derivation, we have divided the general TS fuzzy controllers into several more mathematically tractable classes, each of which is typical and important itself. Specifically, we have studied the TS fuzzy PI controllers that employ just two triangular input fuzzy sets for each input variable, the simplified linear TS rule scheme that we recently introduced, product fuzzy logic OR operator and the centroid defuzzifier [29]. Also, we have investigated the TS fuzzy controllers, the PD type in particular, that use the simplified linear TS rule scheme and Zadeh fuzzy logic AND operator [30]. Moreover, we have extended the derivation to the TS fuzzy PI controllers with the same settings as those in [29] except the simplified linear TS rule scheme is replaced by the original linear TS rule scheme [6]. Our results show all these classes of fuzzy controllers to be nonlinear controllers with variable gains, including nonlinear PI, PD and PID controllers with variable gains. In present paper, we extend our analytical investigation to cover a more general and typical but also more difficult class of TS fuzzy controllers that use at least three trapezoidal or triangular input fuzzy sets, fuzzy rules with linear consequent, Zadeh fuzzy logic AND operator and the centroid defuzzifier. We choose this configuration because it represents typical TS fuzzy controllers in the literature. We will prove these fuzzy controllers to be inherently nonlinear gain scheduling PI or PD controllers.

In conventional control, the purpose of gain scheduling is to connect a number of linear controllers, each of which functions around a different operating point, to form a global controller that is stable and functional not only at these operating points but also along the whole operating trajectory [16]. Several research groups have already studied the relationship between some particular TS fuzzy controllers (not those in this paper though) and the gain scheduling controllers (e.g., [15,18]). All the studies were conducted without knowing the explicit structure of the fuzzy controllers involved. The primary ideas behind these investigations were: (1) each of the TS rule consequent represented a linear (PID) controller and its constant gains were different from rule to rule and hence were different in different regions of input space; and (2) the role of the fuzzy inference mechanism was to make a continuous and smooth control transition between the regions. The studies revealed, at general and conceptual level, the connection between the fuzzy controllers and the gain scheduling controllers. Part of the results, however, was somewhat speculative in nature and not mathematically rigorous due to the lack of explicit structures of the fuzzy controllers. For example, mathematical expression of the control transition between the regions was not derived. Furthermore, according to our results in this paper, unlike conventional gain scheduling controllers, the gains of the nonlinear gain scheduling controllers, naturally realized by the TS fuzzy controllers, constantly vary with the output of the system under control even in the same region. This indicates that the prior results, while important, are incomprehensive to a large extent.

In the second part of this paper, we borrow the well-established Small Gain Theorem from nonlinear control theory to analyze the global bounded-input bounded-output (BIBO) stability of the control systems consisting of the TS fuzzy controllers and nonlinear systems. Local stability is also examined. We effectively used the BIBO stability analysis technique for the Mamdani fuzzy control systems [5]. This BIBO approach is different from the other approaches in the literature (e.g., [19,20,22]) where the Lyapunov methods are utilized for global asymptotic stability analysis of fuzzy control systems without knowing the explicit structure of the TS fuzzy controllers involved. Because our analysis directly uses the controllers' structure, the stability criteria developed have the potential to be less conservative. On the other hand, however, BIBO stability is less informative than asymptotic stability. This pitfall, nevertheless, is partially compensated by our local stability determination that is also in asymptotic sense.

## 2. Configuration of the typical TS fuzzy controllers

The typical TS fuzzy controllers studied in this paper have two inputs and one output. The input variables are error and rate change of error (rate, for short) of system output with respect to output setpoint. They are designated as follows:

$$e(nT) = \mathrm{SP}(nT) - y(nT),$$
$$r(nT) = e(nT) - e(nT - T),$$

where $n$ is a positive integer, $T$ sampling time and $\mathrm{SP}(nT)$ reference/setpoint signal of the system output. We denote $e(nT)$, $r(nT)$ and $y(nT)$ as error, rate and system output, respectively. The two input variables are fuzzified by the same input fuzzy sets. Assume there are $J$ ($J \geqslant 1$) fuzzy sets for positive $e(nT)$ ($r(nT)$), $J$ fuzzy sets for negative $e(nT)$ ($r(nT)$) and one fuzzy set for nearly zero $e(nT)$ ($r(nT)$). Therefore, there are total

$$N = 2J + 1$$

fuzzy sets for $e(nT)$ and $r(nT)$. A fuzzy set for $e(nT)$ (or $r(nT)$) is denoted as $E_i$ (or $R_j$) and the corresponding membership is designated as $\mu_i(e)$ (or $\mu_j(r)$). Fig. 1 illustrates the definitions of the input fuzzy sets over $(-\infty, \infty)$. They are the

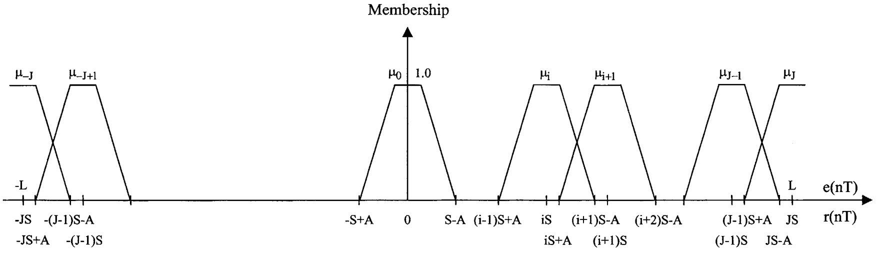


Fig. 1. Illustrative definitions of $N = 2J + 1$ trapezoidal input fuzzy sets for $e(nT)$ and $r(nT)$. Note that $2A$ and $2S$ are respectively the upper side and lower side of each input fuzzy set.

commonly used trapezoidal fuzzy sets, which contain the triangular fuzzy sets, another widely used type, as special cases if $A = 0$ (note that $2A$ is upper side of the trapezoidal fuzzy sets). The lower base is $2S$. The shape of the fuzzy sets is identical to each other and the fuzzy sets are uniformly distributed over $[-L, L]$, where $L$ is a design parameter. Using $x(nT)$ to represent either $e(nT)$ or $r(nT)$, the mathematical definition of $\mu_i(x)$, where $-J < i < J$, is as follows:

$$\mu_i(x) = \begin{cases} 0, & x(nT) \in (-\infty, (i-1)S + A], \\ \frac{x(nT)-(i-1)S-A}{S-2A}, & x(nT) \in ((i-1)S + A, iS - A], \\ 1, & x(nT) \in (iS - A, iS + A], \\ \frac{(i+1)S-A-x(nT)}{S-2A}, & x(nT) \in (iS + A, (i+1)S - A], \\ 0, & x(nT) \in ((i+1)S - A, \infty). \end{cases}$$

When $i = J$ or $-J$, the mathematical definition of $\mu_i(x)$ is respectively as follows:

$$\mu_J(x) = \begin{cases} 0, & x(nT) \in (-\infty, (J-1)S + A], \\ \frac{x(nT)-(J-1)S-A}{S-2A}, & x(nT) \in ((J-1)S + A, JS - A], \\ 1, & x(nT) \in (JS - A, \infty), \end{cases}$$

$$\mu_{-J}(x) = \begin{cases} 1, & x(nT) \in (-\infty, -JS + A], \\ \frac{-(J-1)S-A-x(nT)}{S-2A}, & x(nT) \in (-JS + A, -(J-1)S - A], \\ 0, & x(nT) \in (-(J-1)S - A, \infty). \end{cases}$$

It is obvious that

$$\mu_i(x) + \mu_{i+1}(x) = 1, \quad x \in (-\infty, +\infty).$$

$N^2$ fuzzy control rules are used to cover all the $N \times N$ possible combinations of the input fuzzy sets. The fuzzy controllers under this investigation employ the following form of TS fuzzy rules that use line consequent:

IF $e(nT)$ is $E_i$ AND $r(nT)$ is $R_j$ THEN $\Delta u(nT) = a_{i,j}e(nT) + b_{i,j}r(nT)$

where $\Delta u(nT)$ is the contribution of this rule to the change of the fuzzy controller output, and $a_{i,j}$ and $b_{i,j}$ $(i, j = -J, -J+1, \ldots, J-1, J)$ are design parameters. In the rules, Zadeh fuzzy logic AND operator is used and the resulting membership for the rule consequent is

$$\mu_{i,j}(\Delta u) = \min(\mu_i(e), \mu_j(r)). \tag{1}$$

The widely used centroid defuzzifier is employed to calculate the combined change to the output of the fuzzy controllers

$$\Delta u(nT) = \frac{\sum_{i=-J}^{J} \sum_{j=-J}^{J} \mu_{i,j}(\Delta u)(a_{i,j}e(nT) + b_{i,j}r(nT))}{\sum_{i=-J}^{J} \sum_{j=-J}^{J} \mu_{i,j}(\Delta u)}. \tag{2}$$

The output of the fuzzy controllers at $nT$ is

$$u(nT) = u(nT - T) + \Delta u(nT).$$

## 3. Main results

We now analytically derive the structure of the TS fuzzy controllers and relate the resulting structure to nonlinear PI control as well as gain scheduling control. We then analyze the characteristics of the gains. Based on the derived structure, we analyze the BIBO stability of the TS fuzzy control systems using the Small Gain Theorem. Local stability will also be studied.

### 3.1. Structural analysis of the TS fuzzy controllers

**Theorem 1.** *The TS fuzzy controllers are nonlinear PI controllers with variable proportional-gain and integral-gain.*

**Proof.** We first prove the theorem when both $e(nT)$ and $r(nT)$ are within $[-L, L]$. At any sampling time $nT$, the input variables must satisfy

$$iS \leqslant e(nT) \leqslant (i+1)S$$

and

$$jS \leqslant r(nT) \leqslant (j+1)S.$$

After fuzzification, only the memberships for fuzzy sets $E_i$, $E_{i+1}$, $R_j$ and $R_{j+1}$ are nonzero. Memberships for all the other fuzzy sets are zero. Consequently, only the following four fuzzy control rules are executed:

r1: IF $e(nT)$ is $E_{i+1}$ AND $r(nT)$ is $R_{j+1}$ THEN $\Delta u(nT) = a_{i+1,j+1}e(nT) + b_{i+1,j+1}r(nT)$,
r2: IF $e(nT)$ is $E_{i+1}$ AND $r(nT)$ is $R_j$ THEN $\Delta u(nT) = a_{i+1,j}e(nT) + b_{i+1,j}r(nT)$,
r3: IF $e(nT)$ is $E_i$ AND $r(nT)$ is $R_{j+1}$ THEN $\Delta u(nT) = a_{i,j+1}e(nT) + b_{i,j+1}r(nT)$,
r4: IF $e(nT)$ is $E_i$ AND $r(nT)$ is $R_j$ THEN $\Delta u(nT) = a_{i,j}e(nT) + b_{i,j}r(nT)$.

From (2), we obtain

$$\begin{aligned}\Delta u(nT) &= \frac{\sum_{k=0}^{1}\sum_{m=0}^{1}\mu_{i+k,j+m}(\Delta u)(a_{i+k,j+m}e(nT) + b_{i+k,j+m}r(nT))}{\sum_{k=0}^{1}\sum_{m=0}^{1}\mu_{i+k,j+m}(\Delta u)} \\ &= K_{\mathrm{I}}(e,r)e(nT) + K_{\mathrm{P}}(e,r)r(nT), \end{aligned} \tag{3}$$

where

$$K_{\mathrm{P}}(e,r)=\frac{\sum_{k=0}^{1}\sum_{m=0}^{1}\mu_{i+k,j+m}(\Delta u)b_{i+k,j+m}}{\sum_{k=0}^{1}\sum_{m=0}^{1}\mu_{i+k,j+m}(\Delta u)},$$

$$K_{\mathrm{I}}(e,r)=\frac{\sum_{k=0}^{1}\sum_{m=0}^{1}\mu_{i+k,j+m}(\Delta u)a_{i+k,j+m}}{\sum_{k=0}^{1}\sum_{m=0}^{1}\mu_{i+k,j+m}(\Delta u)}.$$

Recall that the linear PI controller in incremental form is

$$\Delta u(nT)=\overline{K}_{\mathrm{i}}e(nT)+\overline{K}_{\mathrm{p}}r(nT), \tag{4}$$

where $\overline{K}_{\mathrm{p}}$ and $\overline{K}_{\mathrm{i}}$ are proportional-gain and integral-gain, respectively. Comparing (3) with (4), one sees that the TS fuzzy controllers are actually nonlinear PI controllers with variable proportional-gain, $K_{\mathrm{P}}(e,r)$, and integral-gain, $K_{\mathrm{I}}(e,r)$, which are determined by $e(nT)$, $r(nT)$ and the design parameters in the rule consequent (see [27] for more general but less specific results). We call $K_{\mathrm{P}}(e,r)$ and $K_{\mathrm{I}}(e,r)$ dynamic proportional-gain and integral-gain, respectively, because they change with $e(nT)$ and $r(nT)$.

The proof is similar for the cases when either $e(nT)$ or $r(nT)$ is outside $[-L,L]$. □

In the analysis so far, we have related the structure of the TS fuzzy controllers to that of the PI controller. One of the important differences between them is the gains: the linear PI controller uses constant gains whereas the nonlinear PI controllers realized through the fuzzy controllers use variable gains. To analyze the characteristics of the variable gains, we need the explicit expressions of $K_{\mathrm{P}}(e,r)$ and $K_{\mathrm{I}}(e,r)$, and we now derive them.

We first derive the gains when both $e(nT)$ and $r(nT)$ are inside $[-L,L]$. To determine the results of Zadeh fuzzy logic AND operation (1) for rules r1–r4, we must use the square configured by $[iS,(i+1)S]$ and $[jS,(j+1)S]$, as shown in Fig. 2. We also must divide this square into 12 different input combinations (ICs). These divisions are necessary because they will result in, in each of the 12 ICs, a unique inequality between $e(nT)$ and $r(nT)$ when each of the four fuzzy rules is evaluated by Zadeh fuzzy logic AND operator. After applying defuzzification algorithm (2) to each of the 12 resulting memberships [31], we obtain the mathematical structures for $K_{\mathrm{P}}(e,r)$ and $K_{\mathrm{I}}(e,r)$, which are identical, as shown in Table 1. The coefficients $A_1$ to $E$ in the table are given in Table 2. Note that although the mathematical structures for $K_{\mathrm{P}}(e,r)$ and $K_{\mathrm{I}}(e,r)$ are the same, their expressions are actually different because their coefficients are different. In the table, we simplify the expressions significantly by introducing the following notations:

$$\theta=\tfrac{1}{2}(S-2A),$$

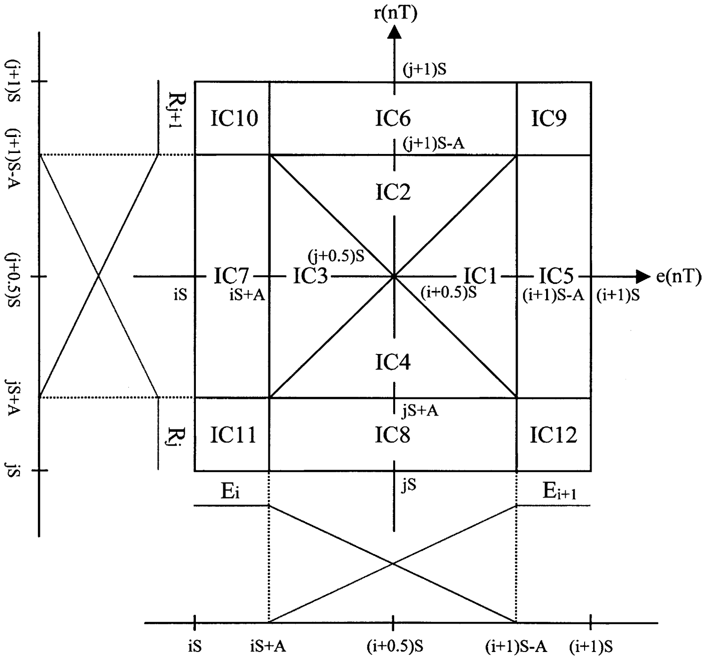


Fig. 2. Division of input space for analytically deriving structure of the TS fuzzy controllers when both $e(nT)$ and $r(nT)$ are within the interval $[-L, L]$.

Table 1
Proportional-gain $K_P(e,r)$ and integral-gain $K_I(e,r)$ of the TS fuzzy controllers when $e(nT)$ and $r(nT)$ are in different ICs as shown in Figs. 2 and 3

| IC# | $K_P(e,r) =$ and $K_I(e,r) =$ |
|---|---|
| IC1 and IC3 | $\frac{(\theta - \lvert\Delta e(nT)\rvert)A_1 + (\theta \pm \Delta r(nT))A_2 + (\theta \mp \Delta r(nT))A_3}{2(2\theta - \lvert\Delta e(nT)\rvert)}$ |
| IC2 and IC4 | $\frac{(\theta - \lvert\Delta r(nT)\rvert)B_1 + (\theta \pm \Delta e(nT))B_2 + (\theta \mp \Delta e(nT))B_3}{2(2\theta - \lvert\Delta r(nT)\rvert)}$ |
| IC5, IC5′, IC7 and IC7′ | $\frac{(\theta \pm \Delta r(nT))C_1 + (\theta \mp \Delta r(nT))C_2}{2\theta}$ |
| IC6, IC6′, IC8 and IC8′ | $\frac{(\theta \pm \Delta e(nT))D_1 + (\theta \mp \Delta e(nT))D_2}{2\theta}$ |
| IC9, IC9′, IC10, IC10′, IC11, IC11′, IC12 and IC12′ | $E$ |

Table 2
The coefficients used in proportional-gain $K_P(e,r)$ and integral-gain $K_I(e,r)$ shown in Table 1

| IC# | $K_P(e,r)$ | | | $K_I(e,r)$ | | |
|---|---|---|---|---|---|---|
| | $A_1$ | $A_2$ | $A_3$ | $A_1$ | $A_2$ | $A_3$ |
| IC1 | $b_{i,j+1}+b_{i,j}$ | $b_{i+1,j+1}$ | $b_{i+1,j}$ | $a_{i,j+1}+a_{i,j}$ | $a_{i+1,j+1}$ | $a_{i+1,j}$ |
| IC3 | $b_{i+1,j+1}+b_{i+1,j}$ | $b_{i,j+1}$ | $b_{i,j}$ | $a_{i+1,j+1}+a_{i+1,j}$ | $a_{i,j+1}$ | $a_{i,j}$ |
| | $B_1$ | $B_2$ | $B_3$ | $B_1$ | $B_2$ | $B_3$ |
| IC2 | $b_{i+1,j}+b_{i,j}$ | $b_{i+1,j+1}$ | $b_{i,j+1}$ | $a_{i+1,j}+a_{i,j}$ | $a_{i+1,j+1}$ | $a_{i,j+1}$ |
| IC4 | $b_{i+1,j+1}+b_{i,j+1}$ | $b_{i+1,j}$ | $b_{i,j}$ | $a_{i+1,j+1}+a_{i,j+1}$ | $a_{i+1,j}$ | $a_{i,j}$ |
| | $C_1$ | $C_2$ | | $C_1$ | $C_2$ | |
| IC5 and IC5′ | $b_{i+1,j+1}$ | $b_{i+1,j}$ | | $a_{i+1,j+1}$ | $a_{i+1,j}$ | |
| IC7 and IC7′ | $b_{i,j+1}$ | $b_{i,j}$ | | $a_{i,j+1}$ | $a_{i,j}$ | |
| | $D_1$ | $D_2$ | | $D_1$ | $D_2$ | |
| IC6 and IC6′ | $b_{i+1,j+1}$ | $b_{i,j+1}$ | | $a_{i+1,j+1}$ | $a_{i,j+1}$ | |
| IC8 and IC8′ | $b_{i+1,j}$ | $b_{i,j}$ | | $a_{i+1,j}$ | $a_{i,j}$ | |
| | $E$ | | | $E$ | | |
| IC9 (IC9′) | $b_{i+1,j+1}(b_{J,J})$ | | | $a_{i+1,j+1}(a_{J,J})$ | | |
| IC10 (IC10′) | $b_{i,j+1}(b_{-J,J})$ | | | $a_{i,j+1}(a_{-J,J})$ | | |
| IC11 (IC11′) | $b_{i,j}(b_{-J,-J})$ | | | $a_{i,j}(a_{-J,-J})$ | | |
| IC12 (IC12′) | $b_{i+1,j}(b_{J,-J})$ | | | $a_{i+1,j}(a_{J,-J})$ | | |

$$\Delta e(nT) = (i+0.5)S - e(nT),$$
$$\Delta r(nT) = (j+0.5)S - r(nT).$$

The geometrical meaning of the notations is as follows. The size of the square in which IC1–IC4 belong to is $2\theta \times 2\theta$. $\Delta e(nT)$ ($\Delta r(nT)$) is the distance, along the $e(nT)$ ($r(nT)$) axis, between the current state, ($e(nT)$, $r(nT)$), and the center of the square, $((i+0.5)S, (j+0.5)S)$, in which the state is in. The use of the notations will make analysis easier and clearer. For instance, from Table 1, one can see that $0 \leqslant \theta \pm \Delta e(nT) \leqslant 2\theta$ and $0 \leqslant \theta \pm \Delta r(nT) \leqslant 2\theta$. Thus, if all the design parameters are positive, $K_P(e,r)$ and $K_I(e,r)$ will always be positive no matter how they will change with $e(nT)$ and $r(nT)$.

We then derive $K_P(e,r)$ and $K_I(e,r)$ when either $e(nT)$ or $r(nT)$ is outside $[-L, L]$. When this is the case, we need to divide the input space outside the square configured by $[-L, L] \times [-L, L]$ into eight ICs, namely IC5′–IC12′, as shown in Fig. 3. The expressions for $K_P(e,r)$ and $K_I(e,r)$ in different ICs can analytically be derived using the same method described above. The results are given in Table 1 with the coefficients being given in Table 2. According to Table 1, when $e(nT)$ and $r(nT)$ are in IC5′–IC8′, the expressions for $K_P(e,r)$ and $K_I(e,r)$ are the same as those in IC5 and IC8.

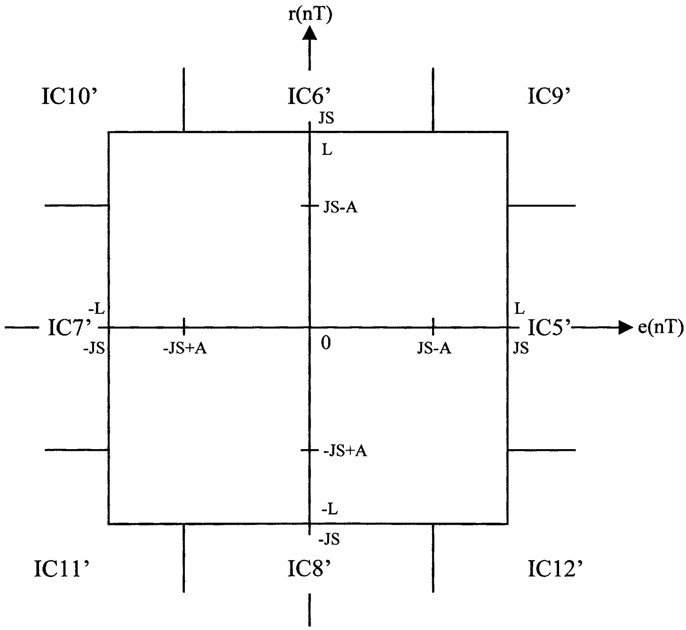


Fig. 3. Division of input space for analytically deriving structure of the TS fuzzy controllers when either $e(nT)$ or $r(nT)$ is outside the interval $[-L, L]$.

Having analyzed the variable gains of the TS fuzzy controllers, let us examine their structure. According to Tables 1 and 2, when $e(nT)$ and $r(nT)$ are in IC1–IC8, a fuzzy controller is a nonlinear PI controller with variable gains. In IC9–IC12 and IC9′–IC12′, a fuzzy controller becomes a linear PI controller with constant gains. A special situation is that $-A \leqslant e(nT) \leqslant A$ and $-A \leqslant r(nT) \leqslant A$. In this case, a fuzzy controller is a linear PI controller with proportional-gain $b_{0,0}$ and integral-gain $a_{0,0}$. Finally, we point out that while the TS fuzzy controllers switch from one PI control algorithm in one IC to another one in another IC when $e(nT)$ and $r(nT)$ vary, the switching is always continuous and smooth on the boundaries of any adjacent ICs involved. In other words, the output of the fuzzy controllers does not change abruptly between different ICs.

Triangular fuzzy sets (i.e., $A = 0$) are special cases of trapezoidal ones. When both $e(nT)$ and $r(nT)$ are in $[-L, L]$ and $A = 0$, IC5–IC12 will not exist anymore, nor will the corresponding linear or nonlinear PI controllers. The value of $A$ does not effect the structural results for the cases when either $e(nT)$ or $r(nT)$ is outside $[-L, L]$.

The values of all the design parameters $a_{i,j}$ and $b_{i,j}$, $i, j = -J, -J + 1, \ldots, J - 1, J$, should not be negative. This is because a negative value for any one of these parameters will result in a negative gain for the corresponding linear PI controller and possibly cause negative gain(s) for the nonlinear PI controllers. A negative gain means a positive feedback controller. Although such a controller will be local (i.e., in $[iS,(i+1)S]\times[jS,(j+1)S]$), it could make the fuzzy control system unstable, and hence should be avoided.

We now analyze the characteristics of the gain variation.

**Lemma 1.** *In* $[iS,(i+1)S]\times[jS,(j+1)S]$ *for all the values of i and j,* $K_P(e,r)$ *and* $K_I(e,r)$ *monotonically increase or decrease in every IC.*

**Proof.** For brevity, we will prove this lemma using IC1 and the proof for the rest of the ICs is similar. At any sampling time $nT$, $e(nT)$ and $r(nT)$ in IC1 satisfy the following:

$$(i+0.5)S \leqslant e(nT) \leqslant (i+1)S - A$$

and

$$jS + A \leqslant r(nT) \leqslant (j+1)S - A.$$

The variable proportional-gain in IC1 is

$$K_P(e,r) = \frac{(\theta - |\Delta e(nT)|)A_1 + (\theta - \Delta r(nT))A_2 + (\theta + \Delta r(nT))A_3}{2(2\theta - |\Delta e(nT)|)}$$

and we have

$$\frac{\partial K_P(e,r)}{\partial r(nT)} = \frac{A_2 - A_3}{2(2\theta - |\Delta e(nT)|)} = \frac{b_{i+1,j+1} - b_{i+1,j}}{2(2\theta - |\Delta e(nT)|)}.$$

If $b_{i+1,j+1} \neq b_{i+1,j}$, $\partial K_P(e,r)/\partial r(nT) \neq 0$ and hence $K_P(e,r)$ does not have any extreme point. In other words, $K_P(e,r)$ is monotonic in IC1. Now, consider $b_{i+1,j+1} = b_{i+1,j}$. When this is the case,

$$K_P(e,r) = \frac{(\theta - |\Delta e(nT)|)A_1 + 2\theta A_2}{2(2\theta - |\Delta e(nT)|)} = \frac{A_1}{2} + \frac{\theta(2A_2 - A_1)}{2(2\theta - |\Delta e(nT)|)},$$

which is monotonic when $|\Delta e(nT)|$ changes from 0 to $\theta$. Combining these two cases, one sees that $K_P(e,r)$ does not have any extreme point and is always monotonic in IC1. The same can be said for IC2–IC4.

In IC5–IC12, $K_P(e,r)$ is of plane. Therefore, $K_P(e,r)$ is monotonic.

As $K_P(e,r)$ and $K_I(e,r)$ have the same mathematical structure, $K_I(e,r)$ is monotonic in IC1–IC12, too. □

**Theorem 2.** *In* $[iS,(i+1)S]\times[jS,(j+1)S]$ *for all the values of i and j,*

(1) $K_P(e,r)$ *achieves its maximum in the whole region of one of IC*9*–IC*12, *and minimum in one of the remaining three ICs apart from the IC region in which* $K_P(e,r)$ *achieves its maximum. The same is true for* $K_I(e,r)$; *and*

(2) *$K_P(e,r)$ and $K_I(e,r)$ satisfy the following inequalities*:

$$\min\{b_{i+1,j+1}, b_{i+1,j}, b_{i,j+1}, b_{i,j}\} \leqslant K_P(e,r) \leqslant \max\{b_{i+1,j+1}, b_{i+1,j}, b_{i,j+1}, b_{i,j}\},$$

$$\min\{a_{i+1,j+1}, a_{i+1,j}, a_{i,j+1}, a_{i,j}\} \leqslant K_I(e,r) \leqslant \max\{a_{i+1,j+1}, a_{i+1,j}, a_{i,j+1}, a_{i,j}\}.$$

**Proof.** We first prove part (1) of the theorem. According to Lemma 1, because of monotonicity, $K_P(e,r)$ in IC1–IC4 can achieve its maximum or minimum only when $(\Delta e(nT), \Delta r(nT))$ is at the following five coordinates: $(0,0)$, $(\theta,\theta)$, $(-\theta,\theta)$, $(\theta,-\theta)$, and $(-\theta,-\theta)$. At $(0,0)$,

$$K_P((i+0.5)S,(j+0.5)S) = \frac{b_{i+1,j+1} + b_{i+1,j} + b_{i,j+1} + b_{i,j}}{4}.$$

At the remaining four coordinates, $K_P(e,r)$ is equal to $b_{i+1,j+1}$, $b_{i+1,j}$, $b_{i,j+1}$, or $b_{i,j}$. Thus, the maximum of $K_P(e,r)$ is the largest parameter whereas the minimum is the smallest parameter among $b_{i+1,j+1}$, $b_{i+1,j}$, $b_{i,j+1}$ and $b_{i,j}$.

In IC5–IC8, according to Table 1, $K_P(e,r)$ is of plane and reaches its maximum on one of the boundaries with IC9, IC10, IC11 or IC12 and its minimum on another boundary with the remaining three ICs apart from the boundary with IC9, IC10, IC11 or IC12 on which the maximum of $K_P(e,r)$ is reached. The maximum and minimum of $K_P(e,r)$ are the same as those of $K_P(e,r)$ in IC1–IC4.

In each of IC9–IC12, $K_P(e,r)$ is a constant, that is, $b_{i+1,j+1}$, $b_{i+1,j}$, $b_{i,j+1}$, or $b_{i,j}$. Hence, the maximum and minimum of $K_P(e,r)$ are the largest and smallest parameters among these four parameters.

Combining these analyses for IC1–IC12, we conclude that: (1) the maximum of $K_P(e,r)$ is the largest parameter whereas the minimum is the smallest one among $b_{i+1,j+1}$, $b_{i+1,j}$, $b_{i,j+1}$, and $b_{i,j}$, and (2) the maximum occurs in one of IC9–IC12 and the minimum happens in the remaining three ICs apart from the IC region in which $K_P(e,r)$ achieves its maximum.

Because $K_I(e,r)$ has the same mathematical structure as that of $K_P(e,r)$, we can obtain the same conclusion for $K_I(e,r)$.

The proof of part (2) of theorem immediately follows. □

In order to visualize how $K_P(e,r)$ (and $K_I(e,r)$) changes with $e(nT)$ and $r(nT)$, we provide three-dimensional plots of $K_P(e,r)$ in three different but typical settings of $b_{i+1,j+1}$, $b_{i+1,j}$, $b_{i,j+1}$, and $b_{i,j}$, as shown in Figs. 4–6. In the plots, we use, without loss of generality, $i = 0$, $S = 0.2$ and $A = 0.04$. The values of the design parameters are: (1) $b_{1,1} = 7$, $b_{1,0} = 6$, $b_{0,1} = 8$ and $b_{0,0} = 9$ for Fig. 4; (2) $b_{1,1} = 8$, $b_{1,0} = 6$, $b_{0,1} = 7$ and $b_{0,0} = 9$ for Fig. 5; and (3) $b_{1,1} = 8$, $b_{1,0} = 7$, $b_{0,1} = 9$ and $b_{0,0} = 6$ for Fig. 6. From these figures, one can see that the

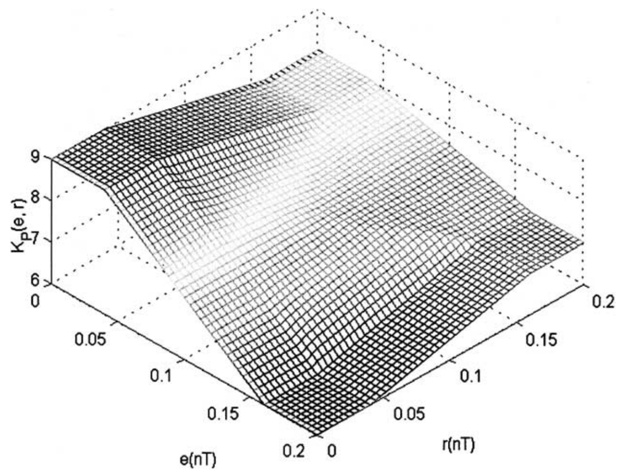


Fig. 4. A three-dimensional plot of proportional-gain, $K_P(e, r)$, of the TS fuzzy controller changing with $e(nT)$ and $r(nT)$ in $[0, 0.2] \times [0, 0.2]$. The values of the design parameters are chosen as: $S = 0.2$, $A = 0.04$, $b_{1,1} = 7$, $b_{1,0} = 6$, $b_{0,1} = 8$ and $b_{0,0} = 9$.

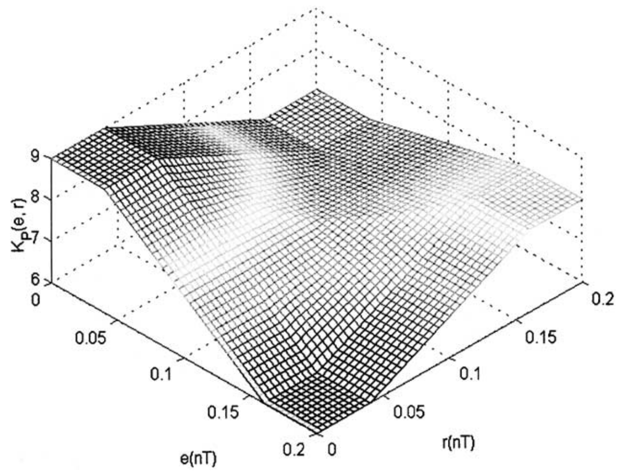


Fig. 5. Another three-dimensional plot of proportional-gain, $K_P(e, r)$, of the TS fuzzy controller changing with $e(nT)$ and $r(nT)$ in $[0, 0.2] \times [0, 0.2]$. The values of the design parameters are the same as those used in Fig. 4 except $b_{1,1} = 8$ and $b_{0,1} = 7$.

maximum and minimum of $K_P(e, r)$ is 9 and 6, respectively. Also can be seen is that in each IC, the gains change monotonically with the change of $e(nT)$ and $r(nT)$. These confirm our theoretical results.

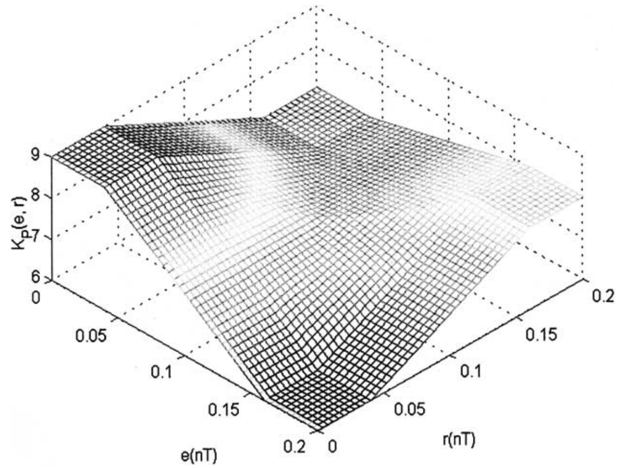


Fig. 6. Another three-dimensional plot of proportional-gain, $K_P(e,r)$, of the TS fuzzy controller changing with $e(nT)$ and $r(nT)$ in $[0, 0.2] \times [0, 0.2]$. The differences between this figure and the two figures above are: $b_{1,1} = 8$, $b_{1,0} = 7$, $b_{0,1} = 9$ and $b_{0,0} = 6$.

These figures demonstrate that one can obtain different characteristics of the gain variation by using different parameter values. For some particular application, the fuzzy controller designer needs to choose proper parameter values to generate the desired gain variation. The variation of $K_I(e,r)$ with respect to $e(nT)$ and $r(nT)$ can also be shown in a similar manner.

Note that the linear PD controller in position form is

$$u(nT) = \overline{K}_p e(nT) + \overline{K}_d r(nT),$$

where $\overline{K}_p$ and $\overline{K}_d$ are the proportional-gain and derivative-gain, respectively. The linear PD controller has the same structure as the linear PI controller (4). Consequently, all the above theorems and analyses can easily be extended to the TS fuzzy controllers that are exactly the same as above ones except in the fuzzy rules, the incremental controller output, $\Delta u(nT)$, is replaced by controller output, $u(nT)$. The resulting TS fuzzy controllers are nonlinear PD controllers with variable gains. The expressions for proportional-gain will be the same as those for $K_I(e,r)$ whereas expressions for derivative-gain will be identical to those for $K_P(e,r)$.

According to the derived structure of the fuzzy controllers as well as the explicit expressions of the gain variation, the TS fuzzy controllers are closely related to the concept of gain scheduling. A TS fuzzy controller is a linear PI controller with constant gains in IC9–IC12, and is a nonlinear PI controller with variable gains in the rest of the ICs. The control transition between any adjacent ICs is always continuous and smooth. For the whole input space, a TS fuzzy controller is actually a nonlinear gain scheduling PI (or PD) controller with variable gains in each IC. Stating this formally, we have

**Theorem 3.** *A typical TS fuzzy controller of the class involved in this paper is a nonlinear gain scheduling PI* (*or PD*) *controller.*

### 3.2. Stability analysis of the TS fuzzy control systems

In this section, we study stability of the TS fuzzy control systems involving nonlinear systems. We first investigate local stability of the fuzzy control systems now. According to Table 1, a TS fuzzy controller becomes a linear PI controller when $e(nT)$ and $r(nT)$ are in $[-A,A]\times[-A,A]$. The proportional-gain is $b_{0,0}$ and integral-gain $a_{0,0}$. Thus, local stability of a TS fuzzy control system in this region can easily be determined by using the linear PI controller and the linearized model of the system under control. Local stability of the TS fuzzy control system is the same as local stability of the linearized fuzzy control system in this region.

We now study global stability of the TS fuzzy control systems using the Small Gain Theorem [8]. We have used the theorem before to derive sufficient BIBO stability conditions for the simple Mamdani fuzzy PI control systems [5] as well as the simple TS fuzzy PI control systems [6]. In what follows, we will use the theorem to analyze the BIBO stability of the TS fuzzy control systems, based on the explicit structure of the TS fuzzy controllers revealed above.

Denote a system, linear or nonlinear, under TS fuzzy control as $P$ and the system output as $P(u(nT))$. We treat the TS fuzzy controller as a nonlinear operator mapping input $e(nT)$ to output $\Delta u(nT)$. Designating the operator as $F$, then $\Delta u(nT)=F(e(nT))$. Because $F$ is different in different $[iS,(i+1)S]\times[jS,(j+1)S]$, we need to discuss stability for every $[iS,(i+1)S]\times[jS,(j+1)S]$. At any sampling time $nT$, the inputs of the TS fuzzy controllers satisfy

$$iS\leqslant e(nT)\leqslant (i+1)S \quad \text{and} \quad jS\leqslant r(nT)\leqslant (j+1)S.$$

We have

$$\begin{aligned}\|\Delta u(e(nT))\| &= \|F(e(nT))\| = \|K_{\mathrm{I}}(e,r)e(nT)+K_{\mathrm{P}}(e,r)r(nT)\|\\ &= \|K_{\mathrm{I}}(e,r)e(nT)+K_{\mathrm{P}}(e,r)(e(nT)-e(nT-T))\| \leqslant (K_{\mathrm{I}}(e,r)\\ &\quad + K_{\mathrm{P}}(e,r))|e(nT)| + K_{\mathrm{P}}(e,r)|e(nT-T)| \leqslant (K_{\mathrm{I}}(e,r)\\ &\quad + K_{\mathrm{P}}(e,r))|e(nT)| + K_{\mathrm{P}}(e,r)e_{\mathrm{M}},\end{aligned}$$

where $e_{\mathrm{M}}$ is the maximum magnitude of the error signal:

$$e_{\mathrm{M}} := \sup_{n\geqslant 1}|e(nT-T)|.$$

In every $[iS,(i+1)S]\times[jS,(j+1)S]$, $e(nT-T)$ is bounded, and so is $e_M$. We should point out that $e_M$ is not only bounded for finite $n$, but also bounded when $n\to\infty$ due to the stability condition in every $[iS,(i+1)S]\times[jS,(j+1)S]$.

According to Theorem 2, we have

$$K_I(e,r)+K_P(e,r)\leqslant\alpha_{i,j}+\beta_{i,j}$$

and

$$K_P(e,r)\leqslant\beta_{i,j},$$

where

$$\alpha_{i,j}=\max\{a_{i+1,j+1},a_{i+1,j},a_{i,j+1},a_{i,j}\}$$

and

$$\beta_{i,j}=\max\{b_{i+1,j+1},b_{i+1,j},b_{i,j+1},b_{i,j}\}.$$

Therefore,

$$\|F(e(nT))\|\leqslant(\alpha_{i,j}+\beta_{i,j})|e(nT)|+\beta_{i,j}e_M,$$

where $iS\leqslant e(nT)\leqslant(i+1)S$.

Note that

$$\|P(u(nT))\|\leqslant\|P\||u(nT)|,$$

where

$$\|P\|:=\sup_{u_1\neq u_2,n\geqslant 0}\frac{|P(u_1(nT))-P(u_2(nT))|}{|u_1(nT)-u_2(nT)|}$$

is the operator norm of a given $P$. The norm is the gain of the given nonlinear system over a set of admissible control signals that have any meaningful function norms [8], and $u_1(nT)$ and $u_2(nT)$ are any two of the control signals in the set. Applying the Small Gain Theorem, we get the sufficient condition for the BIBO stability of the nonlinear TS fuzzy control systems

$$(\alpha_{i,j}+\beta_{i,j})\|P\|<1.$$

In order for the TS fuzzy control systems to be BIBO stable, this stability condition must hold for the whole input space. In other words, it must hold in $[iS,(i+1)S]\times[jS,(j+1)S]$ for all the values of $i$ and $j$. Combining these two conditions, we have the following stability theorem.

**Theorem 4.** *The sufficient condition for the nonlinear TS fuzzy control systems to be BIBO stable is*

$$\max\{\alpha_{i,j}+\beta_{i,j}\}\|P\|<1,\quad \textit{for } i,j=-J,\ldots,J, \tag{5}$$

*where* max{ } *is the operation that chooses the largest value among a set of values.*

Compared to other global stability conditions developed by using the Lyapunov methods but without knowing the analytical structures of the fuzzy controllers, our stability results have the potential to be less conservative because they are derived based on the explicit structure of the fuzzy controllers. On the other hand, however, the BIBO stability, while useful, is less informative than asymptotic stability. This pitfall, nevertheless, is partially compensated by our local stability determination that is also in asymptotic sense.

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