Input-Output Structural Relationship between Fuzzy Controllers Using Nonlinear Input Fuzzy Sets and PI or PD Control

Amin Haj-Ali and Hao Ying

Abstract

Fuzzy controllers are known to be relatively easy to design and implement but hard to give a definite answer about their actual performance (e.g.: local and global stability, performance in the presence of disturbance, etc…) since they are treated as "black-box" controllers (internal system-like interaction or input-output relation is unknown). The analytical structure of fuzzy controllers clears away the difficulties in analyzing the fundamental attributes of fuzzy controllers. The literature contains elaborate structural derivation and analysis for fuzzy controllers with linear fuzzy sets for input variables. The analytical structures for fuzzy controllers that use nonlinear input membership functions are almost unexplored. In this paper, we a give study of the input-output structures of fuzzy controllers with nonlinear input fuzzy sets. Our study focuses on some common fuzzy controllers and gives conditions for their structures to be equivalent to nonlinear PI/PD controllers with variable gains. All the fuzzy controllers use Mamdani type of fuzzy rules, product fuzzy AND operator, singleton output fuzzy sets, and the centroid defuzzifier.

Keywords: Fuzzy Control, Mamdani Fuzzy Controller, Fuzzy Structure Derivation, PI/PD Control.

1. Introduction

Fuzzy controllers are known to be effective in solving practical control problems [13]. Theoretical study and practical applications of fuzzy control have advanced significantly, especially in the past few years [9,12,17]. Many researchers attempted to understand mathematical structures of the fuzzy controllers with various configurations and to analyze and design fuzzy control systems with the aid of the well-developed conventional linear and nonlinear control theories [3,4,10,17]. Recently, significant progress had been made [1,4,8,10]. Input-output structure derivation and precise understanding of the resulting structures in relation to conventional controllers is fundamentally important because without such knowledge, systematic analysis and design will be difficult and ineffective at best, so will be effective utilization of the conventional tools. Fuzzy controllers have been linked to various conventional controllers such as multi-level relay control [15] and sliding mode control (e.g., [7]). The availability of the structural information makes it possible to judge stability of the fuzzy control systems and design stable fuzzy control systems [2,7,11,16], among other benefits.

The literature contains elaborate structural derivation and analysis for fuzzy controllers with linear input membership functions. The analytical structures for fuzzy controllers that use nonlinear input membership functions are almost unexplored. In this paper, we present two types of fuzzy controllers, both of which use nonlinear membership functions for the input variables. The rest of the configuration includes Mamdani fuzzy rules, product fuzzy AND operator, singleton output membership functions, and the centroid defuzzifier. The two controllers differ in the number of input membership functions and the actual rules. The first controller (and what we will refer to as minimal configuration) uses two membership functions for each input variable and four fuzzy rules. The other controller uses multiple membership functions for the input and the so-called linear fuzzy rules.

The goal of this study was to explore the relationship between the fuzzy controllers and the PID control. Specifically, we derived the conditions under which the fuzzy controllers were structurally equivalent to nonlinear PI/PD controllers with variable gains.

2. Minimal Configuration of Fuzzy PI/PD Controllers

The first fuzzy controller that we will evaluate is one of the simplest. It is the same as that in [14] except the fuzzy sets for input variables are nonlinear. It has the following composition:

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1. It drives the plant's output $y(n)$ to follow a single input objective $S(n)$ (reference signal, or set point) by minimizing both the error $e(n)$ and its derivative $r(n)$ where
$$e(n) = S(n) - y(n)$$
$$r(n) = e(n) - e(n-1)$$
$$= y(n-1) - y(n)$$
2. The input fuzzy sets employ nonlinear membership functions.
3. The product fuzzy AND operator is used.
4. The controller is of the Mamdani type, i.e., the rule consequent employ fuzzy sets rather than functions of the input variables.
5. The output variable uses singleton fuzzy sets.
6. The fuzzy controller output is calculated based on the centroid defuzzifier (i.e., the weighted-average defuzzifier).

The input variables $e(n)$ and $r(n)$ are scaled by design parameters K_e and K_r respectively. The scaled input variables $E(n) = K_e e(n)$ and $R(n) = K_r r(n)$ fall in the same universe of discourse $[-L, L]$, where L, a positive number, is a design parameter.

Each input variable is fuzzified by two fuzzy sets, namely "P" (stands for Positive) and "N" (stands for Negative). Both are assumed to be nonlinear, monotonic, and differentiable and are defined over $[-L, L]$. They satisfy the following conditions:

$$\mu_P(e) = 1 - \mu_N(e),$$
$$\mu_P(0) = \mu_N(0) = 0.5,$$
$$\eta_P(r) = 1 - \eta_N(r) \text{ and}$$
$$\eta_P(0) = \eta_N(0) = 0.5. \quad (1)$$

The symbols μ and η will be used to denote a membership function for e and r respectively. The subscripts of μ and η denote the name of the different fuzzy sets. An example is shown in Figure 1. All the membership functions in this paper are assumed to be expandable in Taylor series. For instance,

$$\mu_P(e) = a_{10} + a_{11}K_e e(n) + a_{12}\left(K_e e(n)\right)^2 + a_{13}\left(K_e e(n)\right)^3 + \cdots \quad (2)$$

and

$$\eta_P(r) = a_{20} + a_{21}K_r r(n) + a_{22}\left(K_r r(n)\right)^2 + a_{23}\left(K_r r(n)\right)^3 + \cdots \quad (3)$$

All the commonly used nonlinear membership functions, such as sigmoidal type, bell-shape types, and logistic functions, meet this requirement.

Fuzzy rules for the minimal configuration controllers are described in Table 1. The output variable of the fuzzy controllers is either $u(n)$ (controller output) or $\Delta u(n)$ (change in controller output). It uses three singleton fuzzy sets (see Figure 2), which are labeled as P (for Positive), Z (for Zero), and N (for Negative). Each of them has nonzero membership at only one location of the universes of discourse ($-\infty$, ∞). H is a design parameter.

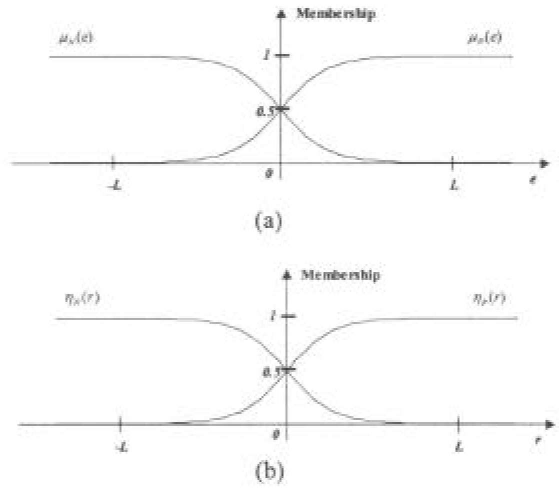

Figure 1. Input fuzzy sets for the minimally-configured fuzzy controller

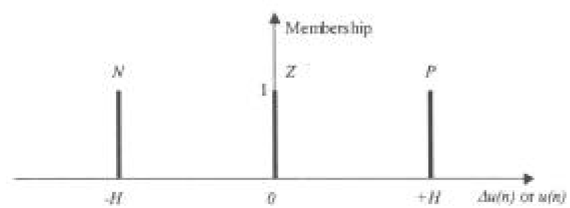

Figure 2. Singleton output fuzzy sets for the minimally-configured fuzzy controller

Table 1. Four fuzzy rules used by the minimal configuration of the fuzzy controller

Rule No.	IF	$E(n)$ is	AND	$R(n)$ is	THEN	$\Delta u(n)$ or $u(n)$ is
1		$\mu_P(e)$		$\eta_P(r)$		P
2		$\mu_P(e)$		$\eta_N(r)$		Z
3		$\mu_N(e)$		$\eta_P(r)$		Z
4		$\mu_N(e)$		$\eta_N(r)$		N

3. Conditions for Fuzzy Controller to be Structurally Nonlinear PI or PD Controller

In this section, we will establish necessary and sufficient conditions for the above configure fuzzy controller to be nonlinear PI or PD controller. Our study will focus on the input space where $E(n)$ and $R(n)$ are in $[-L, L]$.

Lemma 1

$$\Delta u(n) \;\; (\text{or } u(n)) = (\mu_P(e) + \eta_P(r) - 1) \cdot H \tag{4}$$

Proof:

$$\Delta u(n) \;\; (\text{or } u(n)) = \frac{Num}{Den}$$

where:

$$\begin{aligned} Num &= \mu_P(e) \cdot \eta_P(r) \cdot H \\ &\quad + \mu_N(e) \cdot \eta_N(r) \cdot (-H) \\ Den &= \mu_P(e) \cdot \eta_P(r) + \mu_P(e) \cdot \eta_N(r) \\ &\quad + \mu_N(e) \cdot \eta_P(r) + \mu_N(e) \cdot \eta_N(r) \end{aligned} \tag{5}$$

Using the identity of the input membership functions in (1) in the numerator and denominator of (5),

$$\begin{aligned} Num &= \mu_P(e) \cdot \eta_P(r) \cdot H \\ &\quad + \mu_N(e) \cdot \eta_N(r) \cdot (-H) \\ &= \mu_P(e) \cdot \eta_P(r) \cdot H \\ &\quad + (1 - \mu_P(e)) \cdot (1 - \eta_P(r)) \cdot (-H) \\ &= \mu_P(e) \cdot \eta_P(r) \cdot H \\ &\quad - \mu_P(e) \cdot \eta_P(r) \cdot H \\ &\quad + (\mu_P(e) + \eta_P(r)) \cdot H - H \\ &= (\mu_P(e) + \eta_P(r) - 1) \cdot H \end{aligned}$$

and

$$\begin{aligned} Den &= \mu_P(e) \cdot \eta_P(r) \\ &\quad + \mu_P(e) \cdot \eta_N(r) \\ &\quad + \mu_N(e) \cdot \eta_P(r) \\ &\quad + \mu_N(e) \cdot \eta_N(r) \\ &= \mu_P(e) \cdot \eta_P(r) \\ &\quad + \mu_P(e) \cdot (1 - \eta_P(r)) \\ &\quad + (1 - \mu_P(e)) \cdot \eta_P(r) \\ &\quad + (1 - \mu_P(e)) \cdot (1 - \eta_P(r)) \\ &= 1 \end{aligned}$$

thus proving the lemma. Q.E.D.

Theorem 1 A fuzzy controller that uses the input variables $e(n)$ and $r(n)$ with input membership functions satisfying (1), the singleton output fuzzy sets, product fuzzy AND operator, the fuzzy rules as per Table 1, and the centroid defuzzifier structurally becomes a nonlinear PI or PD controller with variable gains in the input space $[-L, L] \times [-L, L]$ if and only if

$$\mu_P(x) = 1 - \eta_P(-x)$$

and

$$\mu_N(x) = 1 - \eta_N(-x) \quad \forall x \in \Re$$

Here, x is either $e(n)$ or $r(n)$.

Proof:

Applying the Taylor expansion of the two membership functions in (2) and (3) to (5), then (5) becomes

$$\begin{aligned} \Delta u(n) \;\; (\text{or } u(n)) = (&-1 + a_{10} + a_{20} \\ &+ a_{11} K_e e(n) \\ &+ a_{21} K_r r(n) \\ &+ a_{12} (K_e e(n))^2 \\ &+ a_{22} (K_r r(n))^2 \\ &+ a_{13} (K_e e(n))^3 \\ &+ a_{23} (K_r r(n))^3 + \cdots) \cdot H \end{aligned} \tag{6}$$

Note that

$$\Delta u(n) = 0 \;\; (\text{or } u(n) = 0), \quad \forall (K_e e(n) + K_r r(n)) = 0$$

Therefore, replacing $K_r r(n)$ by $-K_e e(n)$, (6) will be transformed into a polynomial in $K_e e(n)$:

$$\begin{aligned} \Delta u(n) \;\; (\text{or } u(n)) = (&-1 + a_{10} + a_{20} \\ &+ (a_{11} - a_{21}) K_e e(n) \\ &+ (a_{12} + a_{22}) (K_e e(n))^2 \\ &+ (a_{13} - a_{23}) (K_e e(n))^3 + \cdots) \cdot H \\ \equiv 0 \end{aligned}$$

To realize this equality, the coefficients of all the $K_e e(n)$ terms must be zero simultaneously, leading to

$$\begin{aligned} &a_{10} + a_{20} = 1 \\ &a_{1(2k-1)} = a_{2(2k-1)} \overset{\Delta}{=} a_{2k-1} \\ &a_{1(2k)} = -a_{2(2k)} \overset{\Delta}{=} a_{2k} \end{aligned} \tag{7}$$

for $k = 1, 2, 3, \ldots$

These conditions mean that

$$\mu_P(e) = 1 - \eta_P(-r)$$

and

$$\mu_N(e) = 1 - \eta_N(-r)$$

Replacing all these results back in (6),

$$\begin{aligned}\Delta u(n)\ \ (\text{or } u(n)) &= (a_1(K_e e(n) + K_r r(n)) \\ &+ a_2((K_e e(n))^2 - (K_r r(n))^2) \\ &+ a_3((K_e e(n))^3 + (K_r r(n))^3) + \cdots) \cdot H\end{aligned} \tag{8}$$

The proof is then completed by a direct application of the binomial rules to (8), which allows us to factor out $K_e e(n) + K_r r(n)$ as in (9), with the remaining multiplicand being the variable portion of the variable gains, denoted by $\beta(e,r)$ as given by (10):

$$\begin{aligned}&\Delta u(n)\ (\text{or } u(n)) = \\ &\qquad \beta(e,r) \cdot (K_e e(n) + K_r r(n))\end{aligned} \tag{9}$$

where

$$\begin{aligned}\beta(e,r) = H \cdot (a_1 &+ a_2(K_e e(n) \\ &-K_r r(n)) + a_3((K_e e(n))^2 \\ &-K_e e(n) K_r r(n) \\ &+(K_r r(n))^2) + \theta)\end{aligned} \tag{10}$$

θ represents the error when truncating the series. When $\Delta u(n)$ is used, the controller is a nonlinear PI controller in incremental form with the variable proportional-gain and integral-gain being $\beta(e,r) \cdot K_r$ and $\beta(e,r) \cdot K_e$, respectively. If, instead, $u(n)$ is employed, the controller is a nonlinear PD controller in position form. Its variable proportional-gain and derivative-gain are $\beta(e,r) \cdot K_e$ and $\beta(e,r) \cdot K_r$, respectively. Q.E.D.

4. The Case When the Input Variables Are Fuzzified By More Than Two Fuzzy Sets

Now that we know the conditions for the case of two input fuzzy sets, what will happen if the controller uses more than two membership functions to fuzzify its inputs? To answer this question, we build upon the previous results.

First let us layout the composition of the new controller. The universe of discourse is now covered by N fuzzy sets in each dimension (along the e and r axes). For each input variable, the fuzzy sets are identically shaped, symmetric and evenly distributed along its axis (see Figure 3). At the point of intersection of two membership functions, they both assume the value 0.5. The membership functions still satisfy (1) with the following additional requirement:

$$\mu_{i+1}(e) = 1 - \mu_i(e)$$

and

$$\eta_{i+1}(r) = 1 - \eta_i(e) \tag{11}$$

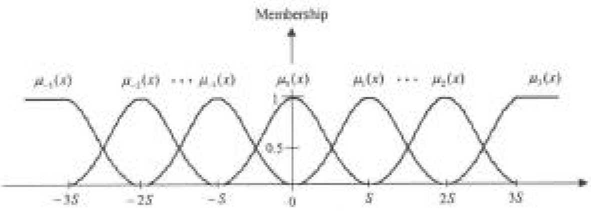

Figure 3. Multiple membership functions for the input variables (case when $N = 7$ and $J = 3$)

The product fuzzy AND operator is still being used. The centroid defuzzifier will always be used to calculate the output. The difference comes in the rules and the output membership functions.

The controller now uses what is known as linear fuzzy rules [15]. The formula of the rules is shown in Table 2. The formula applies to the following region in the $e-r$ space:

$$iS \le e(n) \le (i+1)S$$

$$jS \le r(n) \le (j+1)S$$

where i and j are arbitrary indices of the membership functions ($i = -J,..,J$ and $j = -J,..,J$). The singleton output fuzzy sets distribute uniformly with the space between two consecutive sets being V. There are $2J+1$ of them, ranging from $-2JV$ to $2JV$.

Table 2. Linear fuzzy rules used by the fuzzy controller with multiple membership functions

Rule No.	IF	$E(n)$ is	AND	$R(n)$ is	THEN	$\Delta u(n)$ or $u(n)$ is
1		$\mu_{i+1}(e)$		$\eta_{j+1}(r)$		$(i+j+2)\cdot V$
2		$\mu_{i+1}(e)$		$\eta_j(r)$		$(i+j+1)\cdot V$
3		$\mu_i(e)$		$\eta_{j+1}(r)$		$(i+j+1)\cdot V$
4		$\mu_i(e)$		$\eta_j(r)$		$(i+j)\cdot V$

Theorem 2 A fuzzy controller that uses the input variables $e(n)$ and $r(n)$, the singleton output fuzzy sets, product AND operator, the fuzzy rules as per Table 2, and the centroid defuzzifier becomes the sum of two controllers: (i) a local nonlinear controller that is structurally equivalent to a nonlinear PI or PD controller with variable gains, and (ii) a global multilevel relay action; if and only if the input fuzzy sets satisfy:

$$\mu_{i+1}(x) = 1 - \eta_{j+1}(-x)$$

and

$$\mu_i(x) = 1 - \eta_j(-x) \ \forall i, j \quad (12)$$

Proof:
The resulting control action of this controller is:

$$\Delta u(n) \ (\text{or}\, u(n)) = V \cdot \frac{Num}{Den} \quad (13)$$

where

$$Num = \mu_{i+1}(e)\eta_{j+1}(r)(i+j+2) + \mu_{i+1}(e)\eta_j(r)(i+j+1) + \mu_i(e)\eta_{j+1}(r)(i+j+1) + \mu_i(e)\eta_j(r)(i+j)$$

$$Den = \mu_{i+1}(e)\eta_{j+1}(r) + \mu_{i+1}(e)\eta_j(r) + \mu_i(e)\eta_{j+1}(r) + \mu_i(e)\eta_j(r)$$

Note that the denominator is equal to:

$$Den = \mu_{i+1}(e)\eta_{j+1}(r) + \mu_{i+1}(e)(1-\eta_{j+1}(r)) + (1-\mu_{i+1}(e))\eta_{j+1}(r) + (1-\mu_{i+1}(e))(1-\eta_{j+1}(r))$$

$$Den = \mu_{i+1}(e)\eta_{j+1}(r) + \mu_{i+1}(e) - \mu_{i+1}(e)\eta_{j+1}(r) + \eta_{j+1}(r) - \mu_{i+1}(e)\eta_{j+1}(r) + \eta_{j+1}(r) + 1 - \mu_{i+1}(e) - \eta_{j+1}(r) + \mu_{i+1}(e)\eta_{j+1}(r) = 1$$

Therefore, replacing these results in (13) would give:

$$\Delta u(n) \ (\text{or}\, u(n)) = (\mu_{i+1}(e)\eta_{j+1}(r)(i+j+2) + \mu_{i+1}(e)(1-\eta_{j+1}(r))(i+j+1) + (1-\mu_{i+1}(e))\eta_{j+1}(r)(i+j+1) + (1-\mu_{i+1}(e))(1-\eta_{j+1}(r))(i+j)) \cdot V = V \cdot (\mu_{i+1}(e) + \eta_{j+1}(r) + i + j)$$

It is clear that the resulting control action is the sum of two controls; a local nonlinear PI/PD and a global multilevel relay defined by:

$$\Delta u_{PI}(n) \ (\text{or}\, u_{PD}(n)) = V \cdot (\mu_{i+1}(e) + \eta_{j+1}(r) - 1)$$

one can easily show that the local part is a PI/PD controller by comparing it to the results in Theorem 1. The global relay is given by:

$$\Delta u_{RELAY}(n) \ (\text{or}\, u_{RELAY}(n)) = i + j + 1.$$

Q.E.D.

5. Conclusions

We have established the conditions for a class of fuzzy controllers to be structurally equivalent to a nonlinear PI or PD controller. The resulting conditions were then used as a basis to explore the structure of the same controller but with multiple input fuzzy sets and linear fuzzy rules. Conditions were established for the controller to be the sum of two controllers: (i) a local nonlinear PI/PD controller with variable gain, and (ii) a global multilevel relay. All the conditions are necessary and sufficient.

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