Effects of Increasing the Footprints of Uncertainty on Analytical Structure of the Classes of Interval Type-2 Mamdani and TS Fuzzy Controllers

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Abstract—The fundamental difference between an interval type-2 (IT2) fuzzy controller and a type-1 fuzzy controller is the footprint of uncertainty (FOU) of an IT2 fuzzy set. In this paper, we study how FOUs affect the analytical structure (i.e., the input-output mathematical relationship) of a broad class of IT2 Mamdani and TS controllers. The controllers employ arbitrary fuzzy rules, the KM or EKM type-reducer, the minimum AND operator, and the centroid defuzzifier. The controllers utilize commonly used IT2 fuzzy sets for their input variables and any kind of type-2 fuzzy sets for their output variable. We prove that, with increase of FOUs of the input fuzzy sets, (1) the Mamdani controllers approach constant controllers, and (2) the TS controllers approach piecewise linear controllers. The resemblance to the constant or piecewise linear controllers increases as the FOUs increase. When all the FOUs are at their maximum (to reflect the highest level of uncertainties), the Mamdani and TS controllers become the constant controllers and piecewise linear controllers, respectively. We investigate how change in the resemblance takes place progressively as FOUs increase. We also show that an increase in the resemblance narrows control gain variations for part of the IT2 controllers, which can worsen control performance. These findings implicit narrowing control gain variations for part of the IT2 controllers approach piecewise linear controllers. Real-time control experiment results are provided to illustrate the theoretical analysis.

Index Terms—footprint of uncertainty, interval type-2, Mamdani fuzzy control, TS fuzzy control, KM type-reducer, EKM type-reducer, piecewise control.

I. INTRODUCTION

Developed from interval type-2 (IT2) fuzzy logic [13], IT2 fuzzy control is a relatively new field of interest and research because IT2 fuzzy controllers [5, 7, 10, 11] have the potential to outperform their type-1 (T1) counterparts [1, 2, 4, 6, 9, 22, 30]. An IT2 controller uses IT2 fuzzy sets for its input and/or output variables. Hence, the IT2 fuzzy set is the fundamental factor differentiating an IT2 controller from a T1 controller. A unique feature of an IT2 fuzzy set is a footprint of uncertainty (FOU) that is used to characterize additional uncertainties beyond what a T1 fuzzy set can capture [15, 16, 18, 20, 22, 30].

This paper theoretically studies a key question that has been addressed in the literature mainly through experimental and simulation investigations – how FOU of an IT2 fuzzy set for an input variable impacts analytical structure (i.e., the input-output mathematical relationship) of the fuzzy controller? We believe that revealing the analytical structure of an IT2 fuzzy controller is one of the few ways, if not the only way, to precisely examine how FOU affects IT2 controller’s properties. Such knowledge can lead to an insightful understanding on the role of a FOU plays in control mechanism and performance. We adopt this approach in this paper to probe a wide range of IT2 controllers whose configurations are quite general. Technically speaking, it is more challenging to apply this approach to the IT2 controllers (due to their complexities) than to the corresponding T1 controllers. At the same time, this approach can be very rewarding as evidenced by the numerous analytical structure analysis results produced in the literature on the T1 Mamdani and TS fuzzy controllers (e.g., [25–28]).

At present, the number of results in this direction is relatively small [3, 8, 12, 19, 23, 32–35] (any publication that does not involve the analytical structure of a IT2 controller is not deemed relevant to this study). The results in [35] is particularly relevant because it inspired us to conduct the study presented in the present paper. In that paper, we developed a novel analytical structure-deriving technique for a class of typical IT2 TS fuzzy controllers. With the exposed analytical structures, we proved that under certain conditions, a small subset of two-input-variable IT2 TS fuzzy controllers in the class that employed the simplified linear IT2 rules [35] approached linear or piecewise linear controllers as the FOUs increased. Moreover, these IT2 controllers became linear or piecewise controllers when the FOUs were at their maximum. This interesting finding caused us to ask, and mathematically answer, the following two important questions in the present paper:

1. What will happen to the analytical structure of IT2 Mamdani and TS fuzzy controllers with a quite general configuration when FOUs of their input fuzzy sets become...
larger and larger?

2. What will the analytical structure become when all the FOUs are at their maximum?

To the best of our knowledge, the questions were never raised in the literature before.

As can be seen in Section II, the IT2 fuzzy controllers covered in this study are general and typical. They can use any number of input variables, a broad class of IT2 fuzzy sets for the variables, any type/shape of T2 output fuzzy sets (interval type or not), the KM or EKM type-reducer (mostly widely used type-reducers in the literature), and the centroid defuzzifier. In Section III we mathematically prove that, with the increase of the FOUs of the input fuzzy sets, the following will happen: (1) the general IT2 Mamdani controllers approach constant controllers (which are special cases of piecewise linear controllers), and (2) the general IT2 TS controllers approach piecewise linear controllers. When all the FOUs are at their maximum, the IT2 Mamdani and TS controllers become the constant controllers and piecewise linear controllers, respectively. In Section IV real-time control experiment results are provided to illustrate some of the theoretical analysis. We summarize our findings in Section V.

II. CONFIGURATION OF THE IT2 MAMDANI AND TS FUZZY CONTROLLERS

The IT2 fuzzy controllers in this study have \( n \) input variables, \( x_i(m) \) with \( 1 \leq i \leq n \), and one output variable, \( u(m) \), where \( m \) represents a sampling instance. For simplicity, \( x_1 \) and \( u \) will be used instead of \( x_i(m) \) and \( u(m) \). We suppose that \( x_i \) is in \( [R_i, Q_i] \) that is divided into \( N_i - 1 \) subintervals: \([S_i^1, S_i^2], ..., [S_i^{N_i-1}, S_i^{N_i}]\), \( 1 \leq k \leq N_i \). There are \( N_i \) IT2 fuzzy sets defined over \([R_i, Q_i]\): \( A_i^1, A_i^2, ..., A_i^{N_i} \). \( A_i^k \) is defined over \([S_i^{k-1}, S_i^{k}]\) and \( \theta_i^k \) is used to characterize its footprint of uncertainty (FOU). Note that value of \( \theta_i^k \) can be different for different values of \( x_i \). The upper and lower primary membership functions of \( A_i^k \) are designated as \( \mu_{A_i^k}(x_i, \theta_i^k) \) and \( \mu_{\bar{A}_i^k}(x_i, \theta_i^k) \), respectively. The membership functions can be in any shape as long as they meet the following two requirements: (1) for any specific value of \( x_i \), \( 0 \leq \mu_{A_i^k}(x_i, \theta_i^k) \leq \mu_{\bar{A}_i^k}(x_i, \theta_i^k) \leq 1 \), and (2) in \([S_i^{k-1}, S_i^{k}]\), 

\[
\frac{\partial \mu_{A_i^k}(x_i, \theta_i^k)}{\partial \theta_i^k} \geq 0 \quad \text{and} \quad \frac{\partial \mu_{\bar{A}_i^k}(x_i, \theta_i^k)}{\partial \theta_i^k} \leq 0.
\]

The requirements are not restrictive at all. Indeed, many commonly used fuzzy sets meet them. Fig. 1 shows four example IT2 fuzzy sets (\( A_i^k \) to \( A_i^{k+1} \)) that satisfy the requirements. It should be pointed out that for any given fuzzy controller, it may be not sensible to check the partial derivative because the FOUs of the membership functions will always be fixed (hence the partial derivatives are 0). The requirements make sense when the fuzzy controllers are treated as a class, like what we do in this paper. For each class, there are an infinite number of fuzzy controllers. The components of the fuzzy controllers are the same except for the FOU (i.e., \( \theta_i^k \)) of the membership functions. Increasing or decreasing \( \theta_i^k \) of the membership function of a fuzzy controller in the class will generates a new, different controller of in the same class. In that setting, the partial derivative requirements become reasonable.

![Fig. 1. Four example IT2 fuzzy sets.](image)

A total of \( M = \prod_{i=1}^{n} N_i \) or less fuzzy rules are used to cover all the possible combinations of the input fuzzy sets. This study covers both Mamdani and TS fuzzy controllers. In the case of a Mamdani fuzzy controller, its \( h \)-th rule is (\( h \) is unrelated to the superscripts of the input fuzzy sets)

\[
\text{IF } x_1 = A_i^h \text{ AND } \cdots \text{ AND } x_n = A_i^h \text{ THEN } u = G_s
\]

where \( G_s \) can be any continuous T2 fuzzy set, interval type or not (\( 1 \leq h \leq M \)). The centroid of \( G_s \) is assumed to be \([\alpha_i^h, \beta_i^h]\), where the terminal points are constants. For a TS fuzzy controller, its \( h \)-th rule is

\[
\text{IF } x_1 = A_i^h \text{ AND } \cdots \text{ AND } x_n = A_i^h \text{ THEN } u = c_0^h + c_1^h x_1 + c_2^h x_2 + \cdots + c_n^h x_n
\]

where \( c_0^h \) and \( c_i^h \) are constants.

Zadeh fuzzy AND operator (i.e., \( \min(\cdot) \)) is used in the fuzzy rules. For the \( h \)-th rule, its firing interval, denoted \( f_h(x, \theta) \), is

\[
f_h(x, \theta) = [f_h(x, \theta^1), f_h(x, \theta^2)]
\]

\[
= [\min(\mu_{A_i^h}(x_i, \theta^h_1)), \ldots, \min(\mu_{A_i^h}(x_i, \theta^h_n))],
\]

\[
\min(\mu_{\bar{A}_i^h}(x_i, \theta^h_1), \ldots, \mu_{\bar{A}_i^h}(x_i, \theta^h_n))]
\]

where vector \( x = (x_1, \ldots, x_n) \) and vector \( \theta = (\theta_i^h, \ldots, \theta_i^h) \). For simplicity, \( f_h, f_{\bar{h}} \) and \( f_{\bar{h}} \) will be used instead of \( f_h(x, \theta), f_{\bar{h}}(x, \theta), \) and \( f_{\bar{h}}(x, \theta) \).

The Karnik-Mendel (KM) iterative center-of-sets type-reducer is employed that uses the firing intervals to produce controller output \( u(x, \theta) = [u_1(x, \theta), u_n(x, \theta)] \), which is an interval set (i.e., a special kind of T1 fuzzy set) [13]. \( u_i(x, \theta) \) and \( u_n(x, \theta) \) are the left-most controller output and right-most controller output, respectively. The iterative
operation of the type-reducer requires first arranging the terminal points of the centroids of all the output fuzzy sets in the \( M \) rules in an ascending order. Without loss of generality, assume that the resulting orders for the left and right terminal points of the centroids are \( \alpha_1 \leq \cdots \leq \alpha_s \leq \cdots \leq \alpha_M \) and \( \beta_1 \leq \cdots \leq \beta_s \leq \cdots \leq \beta_M \) (note that \( \alpha_s \) does not necessarily correspond to \( \alpha_s' \); this also holds true for the relationship between \( \beta_j \) and \( \beta_j' \)). One then needs to simultaneously arrange \( f_h \) and \( \overline{f}_h \) in the same order as \( \alpha_1 \leq \cdots \leq \alpha_s \leq \cdots \leq \alpha_M \) to compute \( u_h(x, \theta) \) (for \( u_h(x, \theta) \), simultaneously arrange \( f_h \) and \( \overline{f}_h \) as \( \beta_1 \leq \cdots \leq \beta_s \leq \cdots \leq \beta_M \). Designate the resulting lower and upper terminal points of the centroids \( \overline{f}_h \) and \( \overline{f}_h \), respectively. As for the IT2 TS fuzzy controllers, since there is no centroid as there is no output fuzzy set, alternatively, the type-reducer requires arranging the \( M \) rule consequents in an ascending order according to the magnitudes of their values. Unlike the Mamdani case that has two inequalities, one for \( f_h \) and the other for \( \overline{f}_h \), there will be only one inequality for the TS controllers. Then \( f_h \) and \( \overline{f}_h \) are separately arranged accordingly to the inequality.

For the IT2 Mamdani fuzzy controllers, the KM type-reducer does the following iteratively:

1. Compute \( u_h(x, \theta) = \frac{\sum_{i=1}^{M} f_h i \beta_i}{\sum_{i=1}^{M} f_h i} \) by letting \( f_h i = f_h i + \overline{f}_h i / 2 \).

   Afterward, let intermediate variable \( u_h' = u_h(x, \theta) \).

2. Find such integer \( P_h \), where \( 1 \leq P_h \leq M - 1 \), that \( \beta_{P_h} \leq u_h' \leq \beta_{P_h+1} \).

3. Compute \( u_h(x, \theta) = \frac{\sum_{i=1}^{P_h} f_h i \beta_i}{\sum_{i=1}^{P_h} f_h i} \) with \( f_h i = f_h i \) for \( h \leq P_h \) and \( f_h i = \overline{f}_h i \) for \( h > P_h \). Then, let \( u_h^* = u_h(x, \theta) \) where \( u_h^* \) is another intermediate variable.

4. If \( u_h^* \neq u_h' \), let \( u_h' = u_h^* \) and go back to step (3). Otherwise, let \( u_h(x, \theta) = u_h^* \) and \( u_h(x, \theta) \) is the sought result.

The procedure for computing \( u_h(x, \theta) \) is the same except:

1. subscript \( R \) is replaced by \( L \), \( \beta_j \) by \( \alpha_j \), and \( P_j \) by integer \( P_L \), and (2) let \( f_h = \overline{f}_h \) for \( h \leq P_L \) and \( f_h = \overline{f}_h \) for \( h > P_L \) in step (3). \( P_L \) and \( P_R \) are referred to as switch points whose values depend on the input fuzzy sets, output fuzzy sets, as well as values of the input variables, and hence vary with sampling instance \( m \). The final outcome of the type-reducer is \( u_h(x, \theta) = \frac{\sum_{i=1}^{P_R} f_h i \beta_i + \sum_{i=P_R+1}^{M} f_h i \beta_i}{\sum_{i=1}^{P_R} f_h i + \sum_{i=P_R+1}^{M} \overline{f}_h i} \) \( (2) \) and \( u_L(x, \theta) = \frac{\sum_{i=1}^{P_L} f_h i \alpha_i + \sum_{i=P_L+1}^{M} f_h i \alpha_i}{\sum_{i=1}^{P_L} f_h i + \sum_{i=P_L+1}^{M} \overline{f}_h i} \) \( (3) \).

It should be noted that mathematically speaking, \( u_h(x, \theta) \) and \( u_L(x, \theta) \) represent \( n \)-dimensional surfaces in the space involving the \( n \) input variables. When the FOUs are fixed, they are functions of the input variables.

For the IT2 TS fuzzy controllers, the type-reducer uses the exactly same procedure except replacing both \( \beta_j \) and \( \alpha_j \) by the values of the rule consequents (see Section III-B).

Finally, for either type of fuzzy controllers, the centroid defuzzifier is used to reduce the interval set \( \{ u_L(x, \theta), u_h(x, \theta) \} \) to a number through averaging \( [13] \):

\[
u(x, \theta) = \frac{1}{2} (u_L(x, \theta) + u_h(x, \theta))
\]


II. EFFECTS OF INCREASING FOOTPRINTS OF UNCERTAINTIES ON THE ANALYTICAL STRUCTURE

For better presentation, we will first study the IT2 Mamdani fuzzy controllers in Section A and extend the results to the IT2 TS fuzzy controller in Section B.

A. The IT2 Mamdani Fuzzy Controllers Approach Constant Controllers

As described above, the values of the switch points \( P_h \) and \( P_j \) are between \( 1 \) and \( M - 1 \). The specific values depend on the values of the input variables. We defined a pair of \( P_h \) and \( P_j \) as a Case; every Case was (arbitrarily) assigned a unique number for notation purpose, which was called a Case Number [35]. Points in the input space with the same Case Number that are connected form a region. The entire input space is filled by a (large) number of such regions with different Case Numbers. Mathematical expressions of the boundaries of any two adjacent regions can be explicitly determined through analyzing the type reduction process. For \( P_L \), in step (2) of the type-reducer when \( \alpha_j \leq u_h(x, \theta) \leq \alpha_{j+1} \), \( P_L = j \) \( (1 \leq j \leq M - 1) \). This means that one boundary can be found by solving \( u_h(x, \theta) = \alpha_j \), which divides the input space into two areas. In one area \( P_L < j \) and in the other \( P_L \geq j \). Similarly, solving \( u_h(x, \theta) = \alpha_{j+1} \) creates another boundary. It also divides the input space into two areas. Similar analysis can be conducted for \( P_R \). While the number of Case Numbers may be numerous, it is interesting to point out that when FOUs of the input fuzzy
sets (i.e., \( \theta^i \) where \( i = 1, 2, \ldots, n \)) reach their maximum, there will exist only one Case Number. State formally:

**Lemma 1:** For the IT2 Mamdani fuzzy controllers, when FOU s of all the input fuzzy sets reach their maximum, the number of Case Numbers will reduce to 1. The only Case remains is the Case of \( P_1 = 1 \) and \( P_2 = M - 1 \).

**Proof:** In step (2) of the type-reducer when \( \beta_j \leq u_{\beta_j}(x, \theta) \leq \beta_{j+1}, \) \( P_2 = j \). By the same token, when \( \alpha_j \leq u_{\alpha_j}(x, \theta) \leq \alpha_{j+1}, \) \( P_1 = j \). Combining these inequalities with equations (2) and (3), we have

\[
\begin{align*}
\beta_j & \leq \sum_{h=1}^{j} \frac{f_h}{f_h + \sum_{t=1}^{M-h} f_t} \beta_h + \sum_{h=1}^{M-j} \frac{f_h}{f_h + \sum_{t=1}^{M-h} f_t} \beta_{j+1} \\
& \leq \beta_{j+1} \\
\alpha_j & \leq \sum_{h=1}^{j} \frac{f_h}{f_h + \sum_{t=1}^{M-h} f_t} \alpha_h + \sum_{h=1}^{M-j} \frac{f_h}{f_h + \sum_{t=1}^{M-h} f_t} \alpha_{j+1} \\
& \leq \alpha_{j+1} \cdot j
\end{align*}
\]

Note that when all the FOU s of the input fuzzy sets are at their maximum, the upper and lower fuzzy sets become horizontal lines - Primary Membership = 1 or \( f_h = 1 \) and Primary Membership = 0 or \( f_h = 0 \). Hence (5) becomes

\[
\begin{align*}
\beta_j (M-j) & \leq \beta_{j+1} + \ldots + \beta_M \leq \beta_{j+1}(M-j) \\
\alpha_j \cdot j & \leq \alpha_1 + \ldots + \alpha_j \leq \alpha_{j+1} \cdot j
\end{align*}
\]

In order for the two inequalities to be held, \( j \) must equal 1 for the first equality and \( M = 1 \) for the second inequality. **QED**

Next, we study what happens to \( u_{\beta_j}(x, \theta) \) and \( u_{\alpha_j}(x, \theta) \) when the FOU s of all the input fuzzy sets are at their maximum. Denoting \( u_{\beta_j}(x, \theta) \) and \( u_{\alpha_j}(x, \theta) \) at that time \( U_{\beta_j}(x, \theta) \) and \( U_{\alpha_j}(x, \theta) \) respectively, we have the following result.

**Lemma 2:** For the IT2 Mamdani fuzzy controllers, when FOU s of all the input fuzzy sets reach their maximum, \( U_{\beta_j}(x, \theta) = \beta_M \) and \( U_{\alpha_j}(x, \theta) = \alpha_1 \).

**Proof:** For \( u_{\beta_j}(x, \theta) \) and \( u_{\alpha_j}(x, \theta) \) in equations (2) and (3), the switch points \( P_2 \) and \( P_1 \) vary from 1 to \( M-1 \), resulting in \( M-1 \) different \( u_{\beta_j}(x, \theta) \) and \( u_{\alpha_j}(x, \theta) \). We adopt the following notations:

\[
\begin{align*}
u_{\beta_j}(x, \theta) &= \frac{f_j \beta_j + \ldots + f_1 \beta_1 + \sum_{t=1}^{M-j} f_t \beta_M}{f_j + \ldots + f_1 + \sum_{t=1}^{M-j} f_t} \quad \text{(when } P_2 = j \text{)} \quad (7)
\end{align*}
\]

\[
\begin{align*}
u_{\alpha_j}(x, \theta) &= \frac{f_j \alpha_j + \ldots + f_1 \alpha_1 + \sum_{t=1}^{M-j} f_t \alpha_M}{f_j + \ldots + f_1 + \sum_{t=1}^{M-j} f_t} \quad \text{(when } P_1 = j \text{)} \quad (8)
\end{align*}
\]

When all the FOU s are at their maximum, \( f_h = 1 \) and \( f_j = 0 \) for all \( h \). At that time, \( u_{\beta_j}(x, \theta) = \frac{\alpha_1 + \ldots + \alpha_j}{j} \) and \( u_{\alpha_j}(x, \theta) = \frac{\beta_j + \ldots + \beta_M}{M-j} \), which are two constants. We denote them \( U_{\beta_j}(x, \theta) \) and \( U_{\alpha_j}(x, \theta) \) respectively. According to Lemma 1, \( P_1 = j = 1 \) and \( P_2 = j = M - 1 \). Therefore, \( U_{\beta_j}(x, \theta) = \alpha_1 \) and \( U_{\alpha_j}(x, \theta) = \beta_M \). **QED**

Lemma 2 establishes that: (1) \( u_{\beta_j}(x, \theta) = U_{\beta_j}(x, \theta) = \beta_M \) and \( u_{\alpha_j}(x, \theta) = U_{\alpha_j}(x, \theta) = \alpha_1 \), and (2) \( U_{\beta_j}(x, \theta) \) and \( U_{\alpha_j}(x, \theta) \) are two \( n \)-dimensional hyper-planes that are independent of \( x \) and are characterized by \( \beta_M \) and \( \alpha_1 \). Figure 2 illustrates an example hyper-plane when \( x = (x_1, x_2) \). The plane is parallel to the \( x_1 - x_2 \) plane.

![Figure 2](image)

**Fig. 2:** An illustration of how \( u_{\beta_j}(x, \theta) \) approaches \( U_{\beta_j}(x, \theta) \) for Lemma 3 when \( x = (x_1, x_2) \).

Next, we study the manner in which \( u_{\beta_j}(x, \theta) \) and \( u_{\alpha_j}(x, \theta) \) approach the hyper-planes.

**Lemma 3:** As the FOU s of the input fuzzy sets increase, \( u_{\beta_j}(x, \theta) \) and \( u_{\alpha_j}(x, \theta) \) approach respectively \( U_{\beta_j}(x, \theta) \) and \( U_{\alpha_j}(x, \theta) \) with the following properties:

(a) \( u_{\beta_j}(x, \theta) \leq U_{\beta_j}(x, \theta) \) and \( u_{\alpha_j}(x, \theta) \geq U_{\alpha_j}(x, \theta) \) for any \( x \) when the FOU of at least one of the input fuzzy sets is not at its maximum.

(b) for any \( x \), distances between \( u_{\beta_j}(x, \theta) \) and \( U_{\beta_j}(x, \theta) \) and between \( u_{\alpha_j}(x, \theta) \) and \( U_{\alpha_j}(x, \theta) \), which are functions of \( \theta^i_k \), reduce in a monotonic or non-increasing fashion as \( \theta^i_k \) \( (i = 1, 2, \ldots, n) \) increase.

**Proof:** (a) Let \( \Delta u_{\beta_j}(x, \theta) = U_{\beta_j}(x, \theta) - u_{\beta_j}(x, \theta) \) and \( \Delta u_{\alpha_j}(x, \theta) = u_{\alpha_j}(x, \theta) - U_{\alpha_j}(x, \theta) \). Using (7) and (8) and keeping in mind that \( u_{\beta_j}(x, \theta) \) represents \( u_{\beta_j}(x, \theta) \) when \( P_2 = j \) and \( u_{\alpha_j}(x, \theta) \) indicates \( u_{\alpha_j}(x, \theta) \) when \( P_1 = j \), we obtain

\[
\begin{align*}
\Delta u_{\beta_j}(x, \theta) &= \left\{ \frac{\sum_{h=1}^{M-j} f_h (\beta_M - \beta_h) + \sum_{h=1}^{M-j} f_h (\beta_h - \beta_M)}{f_j + \ldots + f_1 + \sum_{t=1}^{M-j} f_t} \right\} \\
\Delta u_{\alpha_j}(x, \theta) &= \left\{ \frac{\sum_{h=1}^{M-j} f_h (\alpha_1 - \alpha_h) + \sum_{h=1}^{M-j} f_h (\alpha_h - \alpha_1)}{f_j + \ldots + f_1 + \sum_{t=1}^{M-j} f_t} \right\}
\end{align*}
\]
\[ \Delta u_j(x, \theta) = \left( \frac{f'_i(a_i - a_{i-1}) + \cdots + f'_i(a_i - a_j)}{f'_i + \cdots + f'_{i+j} + \cdots + f'_N} \right) \]

Because \( a_i \leq \cdots \leq a_j \leq \cdots \leq a_M \) and \( \beta_i \leq \cdots \leq \beta_j \leq \cdots \leq \beta_M \), as well as \( f'_i \geq 0 \) and \( f'_i \geq 0 \) because the membership values of all the input fuzzy sets are always greater than or equal to 0, \( \Delta u_j(x, \theta) \geq 0 \) and \( \Delta u_j(x, \theta) \geq 0 \). That means \( u_k(x, \theta) \leq U_k(x, \theta) \) and \( u_k(x, \theta) \leq U_k(x, \theta) \).

(b) To prove the distances represented by \( \Delta u_h(x, \theta) \) and \( \Delta u_l(x, \theta) \) to be either decreasing or non-increasing in a monotonic fashion with respect to \( \theta_i^k \) \( (i=1,2,\ldots,n) \) is to prove \( \frac{\partial \Delta u_h(x, \theta)}{\partial \theta} \leq 0 \) and \( \frac{\partial \Delta u_l(x, \theta)}{\partial \theta} \leq 0 \) for any \( x \). For brevity, we will only prove \( \frac{\partial \Delta u_h(x, \theta)}{\partial \theta} \leq 0 \). From (10), we attain

\[ \frac{\partial \Delta u_h(x, \theta)}{\partial \theta} = \frac{\partial}{\partial \theta} \left[ \frac{f'_i(a_i - a_{i-1}) + \cdots + f'_i(a_i - a_h)}{f'_i + \cdots + f'_{i+j} + \cdots + f'_N} \right] \]

(when \( 1 \leq h \leq j \) \( (11) \))

\[ \frac{\partial \Delta u_l(x, \theta)}{\partial \theta} = \frac{\partial}{\partial \theta} \left[ \frac{f'_i(a_i - a_{i-1}) + \cdots + f'_i(a_i - a_j)}{f'_i + \cdots + f'_{i+j} + \cdots + f'_N} \right] \]

(when \( j+1 \leq h \leq M \) \( (12) \))

Note that the denominators of equations (11) and (12) are always positive. Hence what we need to prove is that the numerators are less than or equal to 0. Recall that for \( P_2 \), in step (2) of the type-reducer, when \( a_i \leq u_i(x, \theta) \leq a_{i+1}, P_2=j \). Let \( B_{ij} = u_i(x, \theta) - a_i \) and \( B_{(i+1)} = u_i(x, \theta) - a_{i+1} \). Using equation (3), we obtain

\[ B_{ij} = \left( \frac{f'_i(a_i - a_{i-1}) + \cdots + f'_i(a_i - a_j)}{f'_i + \cdots + f'_{i+j} + \cdots + f'_N} \right) \]

\[ B_{(i+1)} = \left( \frac{f'_{i+1}(a_i - a_{i-1}) + \cdots + f'_{i+1}(a_i - a_{i+1})}{f'_{i+1} + \cdots + f'_{i+j} + \cdots + f'_N} \right) \]

(13)

(14)

Note that \( 0 \leq f'_i \leq f'_i \leq 1 \), as well as \( B_{ij} \geq 0 \) and \( B_{(i+1)} \leq 0 \) due to \( a_i \leq u_i(x, \theta) \leq a_{i+1} \). Thus, from equations (13) and (14) we have the following inequality:

\[ f'_i(a_j - a_{i-1}) + \cdots + f'_i(a_j - a_i) + f'_i(a_i - a_{i-1}) + \cdots + f'_i(a_i - a_j) \]

\[ + f'_{i+1}(a_i - a_{i-1}) + \cdots + f'_{i+1}(a_i - a_{i+1}) \]

\[ + f'_{i+2}(a_{i+1} - a_{i+2}) + \cdots + f'_{i+M}(a_{i+1} - a_M) \]

\[ \geq 0 \]

(15)

Since \( a_k \leq \alpha_i \), for \( 1 \leq h \leq j \) (part of the type-reducer process described in Section II), replacing \( \alpha_i \) in equation (13) by \( \alpha_k \) results in

\[ f'_i(a_k - a_{i-1}) + \cdots + f'_i(a_k - a_i) + f'_i(a_i - a_{i-1}) + \cdots + f'_i(a_i - a_k) \]

\[ + f'_{i+1}(a_i - a_{i-1}) + \cdots + f'_{i+1}(a_i - a_{i+1}) + f'_{i+2}(a_{i+1} - a_{i+2}) + \cdots + f'_{i+M}(a_{i+1} - a_M) \]

\[ \leq 0 \]

(16)

which is the main portion of the numerators of equations (11) and (12). The remaining portion is \( \frac{\partial f'_i(x, \theta)}{\partial \theta} \) for (11) and \( \frac{\partial f'_i(x, \theta)}{\partial \theta} \) for (12). We now show \( \frac{\partial f'_i(x, \theta)}{\partial \theta} \geq 0 \) for (11).

According to equation (1), \( f'_i(x, \theta) = \min(p_{k_i}(x, \theta^i), \ldots, p_{k_i}(x, \theta^i)) \). Therefore, \( \frac{\partial f'_i(x, \theta)}{\partial \theta} \geq 0 \) because \( \frac{\partial p_{k_i}(x, \theta^i)}{\partial \theta} \geq 0 \) as assumed in Section II. Combining these analyses leads to \( \frac{\partial u_k(x, \theta)}{\partial \theta} \leq 0 \) for equation (11). In the same approach, one can prove that \( \frac{\partial u_k(x, \theta)}{\partial \theta} \leq 0 \) for equation (12) as well as \( \frac{\partial u_k(x, \theta)}{\partial \theta} \leq 0 \) for equation (9). Since the value of \( j \) here is arbitrary, these inequalities hold true for all the switch points \( P_2 \) and \( P_N \).

A general function \( g(z_1, \ldots, z_n) \) defined in \( R^s \) is said to be a non-decreasing (or non-increasing) function if for any point \( (z_1^+, z_2^+, \ldots, z_n^+) \) in \( R^s \), \( \frac{\partial g}{\partial z_{i+1}} \left|_{z_{1}^0, z_{2}^0, \ldots, z_{i}^0} \right. \) \( \cdots \) \( \frac{\partial g}{\partial z_{n}^0} \left|_{z_{1}^0, z_{2}^0, \ldots, z_{n}^0} \right. \) are all non-negative (or non-positive) with respect to the respective increase of \( z_1, \ldots, z_n \). Therefore, for any \( x \), \( |u_k(x, \theta) - U_k(x, \theta)| = 0 \) and \( |u_k(x, \theta) - U_k(x, \theta)| = 0 \) when all the FOUs are at their maximum, \( |u_k(x, \theta) - U_k(x, \theta)| = 0 \) and \( |u_k(x, \theta) - U_k(x, \theta)| = 0 \) making \( u_k(x, \theta) = a_k \) owing to Lemma 2.

QED

Lemma 3 reveals the way \( u_k(x, \theta) \) and \( u_l(x, \theta) \) approach their respective hyper-plane. For any \( x \), the approaching is monotonic (or sometimes non-increasing) as \( \theta_i^k \) \( (i=1,2,\ldots,n) \) increase. Fig. 2 provides an example to visualize how \( u_k(x, \theta) \) approaches \( U_k(x, \theta) = a_k \) as the FOUs increase.

Our primary interest is how the controller output \( u(x, \theta) \) changes as the FOUs increase. Knowing the behaviors of
u_i(x,θ) and u_k(x,θ) leads us to achieve the goal because the average of u_i(x,θ) and u_k(x,θ) equals u(θ) (equation (4)). The behavior of u(θ) is characterized as follows:

**Theorem 1:** For the IT2 Mamdani fuzzy controllers, u(x,θ) approaches the constant \( \alpha_i + \beta_iμ \) as the FOUs of the input fuzzy sets increase with the distance being less than \( \frac{1}{2} |u_i(x,θ) - α_1| + \frac{1}{2} |u_k(x,θ) - β_kμ| \). The distance becomes 0, when all the FOUs reach their maximum.

**Proof:**
\[
\frac{|u(x,θ) - U_i(x,θ) + U_k(x,θ)|}{2} = \frac{1}{2} \left| \frac{u_i(x,θ) + u_k(x,θ) - α_1 + β_kμ}{2} \right| \leq \frac{1}{2} \left| u_i(x,θ) - α_1 \right| + \frac{1}{2} \left| u_k(x,θ) - β_kμ \right| \tag{17}
\]

According to Lemma 3, the right side of the inequality approaches 0 in a monotonic decreasing or non-increasing manner with the increase of the FOUs. Hence, \( \frac{|u(x,θ) - \alpha_i + \beta_iμ|}{2} \) will become smaller and smaller as the FOUs become larger and larger. The right side becomes 0 when all the FOUs reach their maximum. At that time, \( u(x,θ) = \alpha_i + \beta_iμ \).

The inequality \( \frac{|u(x,θ) - \alpha_i + \beta_iμ|}{2} \leq \frac{1}{2} \left| u_i(x,θ) - α_1 \right| + \frac{1}{2} \left| u_k(x,θ) - β_kμ \right| \) itself does not automatically rule out the possibility that u(x,θ) approaches \( \frac{\alpha_i + \beta_iμ}{2} \) in a monotonic decreasing or non-increasing fashion. Naturally, one may wonder whether the controller output changes monotonically with respect to the increasing FOUs. What we found is that the change is not monotonic. We made such conclusion through numerical investigation using a MATLAB program that we wrote for this purpose. The result stated in Theorem 1 can be visualized via Fig. 2 if one treats surface \( u_i(x,θ) \) as (equation (4)).

The IT2 TS Fuzzy Controllers Approach Piecewise Linear Controllers

Denote u of TS Rule h in Section II \( u_i \), that is, \( u_i = c_i^h + c_i^h x_1 + c_i^h x_2 + \cdots + c_i^h x_n \). There are M \( u_i \) as there are M TS rules. As mentioned above, in the type reduction process, \( u_i \) must arranged in an ascending order according to the magnitudes of their values, which depends on \( x \) after the constants \( c_i^h \) and \( c_i^h \) are chosen. It is important to keep in mind that different values of \( x \) may or may not cause an ascending order to change. All the different values of \( x \) that correspond to the same ascending order collectively form a region in the \( n \)-dimensional input space. We called such a region a Rule Input Combination (Rule-IC) [35]. The IT2 Mamdani controllers above use the fuzzy sets as rule consequents. Thus, they do not have a Rule-IC, and are relatively simpler to analyze. For the TS controllers, there can be up to \( \sum_1^N \) possible ascending orders. Every Rule-IC has one and only one ascending order. However, one does not know which Rule-IC has which order if the values of \( x \), \( c_i^h \) or \( c_i^h \) are unknown or are not specified, which is the case for this study. Without loss of generality, assume the order for a Rule-IC of interest is \( u_i' \leq \cdots \leq u_i'' \). The corresponding order arrangements for \( f_i' \) and \( f_i'' \) are supposedly \( f_i' \) and \( f_i'' \), respectively. Treat the \( u_i' \leq \cdots \leq u_i'' \) as both \( \alpha_i \leq \cdots \leq \alpha_i \) and \( \beta_i \leq \cdots \leq \beta_i \) of the Mamdani controllers. Keep in mind that there may be more than one Case Number in any one of the Rule-ICs. Then, much of the theoretical work established in Section III-A can be directly extended to the IT2 TS controllers. In light of Lemma 1, the following result is obvious:

**Lemma 4:** For the IT2 TS fuzzy controllers, when the FOUs of all the input fuzzy sets reach their maximum, the number of Case Numbers will reduce to 1 for all the Rule-ICs. The remaining Case Number corresponds to \( P_i = 1 \) and \( P_i = M - 1 \).

**Proof:** One replaces both \( \alpha_i \leq \cdots \leq \alpha_i \) and \( \beta_i \leq \cdots \leq \beta_i \) by \( u_i' \leq \cdots \leq u_i'' \) and then uses the same line of analysis as in the proof of Lemma 1. The conclusion readily follows.

QED

Denote \( u_i(x,θ) \) and \( u_i(x,θ) \) \( U_h\) and \( U_h\), respectively, when the FOUs of all the input fuzzy sets are at their maximum. The following finding is the IT2 TS controllers’ version of Lemma 2 of the Mamdani controllers.

**Lemma 5:** When the FOUs of all the input fuzzy sets are at their maximum, \( u_i(x,θ) \) and \( u_i(x,θ) \) become \( n \)-dimensional hyper-planes in each Rule-IC:

\[
u_i(x,θ) = U_i^h(x,θ) = \Gamma_0 + \Gamma_1 x_1 + \cdots + \Gamma_n x_n \tag{18}
\]

\[
u_i(x,θ) = U_i^h(x,θ) = \Omega_0 + \Omega_1 x_1 + \cdots + \Omega_n x_n \tag{19}
\]

where \( \Gamma_0 \), \( \Gamma_1 \), \( \Gamma_n \) are constants whose values depend on the Rule-IC.

**Proof:** The proof is similar to that for Lemma 2 where it is established that \( U_i(x,θ) = \alpha_i \) and \( U_i(x,θ) = \beta_i \). Note that for a Rule-IC of a TS controller, the role of \( \alpha_i \) is played by \( u_i' \) and \( \beta_i \) by \( u_i'' \) where

\[
u_i' = \Omega_0 + \Omega_1 x_1 + \cdots + \Omega_n x_n \tag{20}
\]
\[ u^*_i = \gamma_0 + \Gamma_z x_1 + \cdots + \Gamma_n x_n \quad (21) \]

\[ \Omega_L, \Omega_R, \Gamma_0 \text{ and } \Gamma_z \text{ are constants whose values depend on Rule-ICs.} \]

The lemma’s conclusion readily follows. QED

The following result is, in a sense, parallel to Lemma 3 of the Mamdani controllers.

**Lemma 6:** For the IT2 TS fuzzy controllers, as the FOUs of the input fuzzy sets become larger, \( u_k(x, \theta) \) and \( u_j(x, \theta) \) approach respectively \( U^{TS}_k(x, \theta) \) and \( U^{TS}_j(x, \theta) \) closer in each Rule-IC with the following properties:

(a) \( u_k(x, \theta) < U^{TS}_k(x, \theta) \) and \( u_j(x, \theta) > U^{TS}_j(x, \theta) \) for any \( x \) in any Rule-IC when the FOU of at least one input fuzzy set is not at its maximum, and

(b) distances \( |u_k(x, \theta) - U^{TS}_k(x, \theta)| \) and \( |u_j(x, \theta) - U^{TS}_j(x, \theta)| \) are either decreasing in a monotonic fashion or non-increasing with the increase of \( \theta^*_i \) \( (i = 1, 2, \ldots , n) \)

**Proof:**

(a) The proof is similar to the proof of part (a) of Lemma 3. Replace both \( \alpha \leq \cdot \cdot \cdot \leq \alpha_j \) and \( \beta \leq \cdot \cdot \cdot \geq \beta_j \) in that proof by \( u_1 \leq \cdot \cdot \cdot \leq u_j \leq \cdot \cdot \cdot \leq u_d \).

(b) The proof is similar to the proof of part (b) of Lemma 3. For any Rule-IC, use \( u_1 \leq \cdot \cdot \cdot \leq u_j \leq \cdot \cdot \cdot \leq u_d \) to replace \( \alpha \leq \cdot \cdot \cdot \leq \alpha_j \) and \( \beta \leq \cdot \cdot \cdot \geq \beta_j \) in the proof of Lemma 3. Thus, for example, for the TS controllers, the equation corresponding to equation (11) is

\[ \frac{\partial u_k}{\partial \theta} = \frac{-\sum_{i=1}^{n} f_i \left[ u_i - u_i^* \right] + \sum_{j=1}^{m} f_j \left[ u_j - u_j^* \right]}{\left( \sum_{i=1}^{n} f_i + \sum_{j=1}^{m} f_j \right)^2} \]

(when \( 1 \leq h \leq j \)) \( (22) \)

By the same token, \( \alpha_j \leq u_k(x, \theta) \leq \alpha_{j+1} \), \( B_{ij} = u_k(x, \theta) - \alpha_j \), and \( B_{i(j+1)} = u_k(x, \theta) - \alpha_{j+1} \) that follow equation (11) in the proof of Lemma 3 need to be replaced respectively by \( u_1 \leq \cdot \cdot \cdot \leq u_j \leq \cdot \cdot \cdot \leq u_d \), \( B_{ij} = u_k(x, \theta) - u_j^* \), and \( B_{i(j+1)} = u_k(x, \theta) - u_{j+1}^* \). Taking the steps along this line that are similar to those taken when Lemma 3 is proved for the TS controllers will lead to the conclusion (b). Fig. 3 provides an example to visualize how \( u_k(x, \theta) \) approaches \( U_k(x, \theta) = \alpha_j \) as the FOUs increase.

QED

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in any Rule-IC. Because the values of the constants $\Omega_0$, $\Omega_1$, $\Gamma_0$, and $\Gamma_1$ depend on Rule-ICs, they may be different for different Rule-ICs. Thus, the planes of all the Rule-ICs form a piecewise linear controller for the entire input space. QED

We make a remark similar to that we make immediately after Theorem 1’s proof. The inequality in (23) might make one wonder whether $u_c(x, \theta)$ approaches the hyper-plane in (24) in a monotonic decreasing or non-increasing fashion. We investigated this possibility through computer simulation and found that this cannot be the case.

It is often easier to design a piecewise linear controller than a nonlinear controller. One implication of Theorem 2 is that one might sometimes want to start with a piecewise linear controller and then use large FOU to construct the IT2 TS controller, which is expected to behave more or less like the piecewise linear controller. Then, gradually decrease the FOU to search for a suitable nonlinear controller for the application in hand.

C When the IT2 Mamdani and TS Fuzzy Controllers Employ the EKM Type-reducer Instead of the KM Type-reducer

The Enhanced KM (EKM) type-reducer uses the following five-step iterative process [12, 24]:

1. Set $h$ to the integer nearest to $M/1.7$ and compute intermediate variables $a = \sum_{j=1}^{H} \beta_j + \sum_{j=M}^{H} \alpha_j \beta_j$ and $b = \sum_{j=1}^{H} \alpha_j \beta_j + \sum_{j=M}^{H} \alpha_j \beta_j$. Let intermediate variable $c' = a / b$.
2. Find such an integer $h' \in [1, M - 1]$ that $\beta_{h'} \leq \beta_{h+1}$.
3. Check whether $h' = h$. If yes, the iteration is over, and set $u_a(x, \theta) = c'$ and $P_a = h$. If no, go to next step.
4. Compute $a' = a - \text{sign}(h'-h) \sum_{j=M}^{H} \beta_j (T_j - T'_j)$ and $b' = b - \text{sign}(h'-h) \sum_{j=M}^{H} (T_j - T'_j)$, where $\text{sign}(h'-h) = 1$ (or -1) when $h' > h$ (or $h' < h$). Compute intermediate variable $c'' = a'/b'$.
5. Set $c' = c''$, $a = a'$, $b = b'$ and $h = h'$. Go to Step 2.

$u_a(x, \theta)$ is computed iteratively in a similar fashion. We show now the use of the EKM algorithm produces the exactly same $u_a(x, \theta)$ and $u_a(x, \theta)$ in equations (2) and (3) as generated by the KM algorithm. The proof is actually quite simple. Note that in step (3) of the EKM algorithm, $u_a(x, \theta) = c'$ where $c' = a/b$ computed in step (1) or in steps (4) and (5). If it is produced in step (1), then $u_a(x, \theta) = c' = \frac{\sum_{j=1}^{H} \beta_j + \sum_{j=1}^{H} \alpha_j \beta_j}{\sum_{j=1}^{H} \alpha_j \beta_j + \sum_{j=1}^{H} \alpha_j \beta_j}$, which is exactly the same as equation (2). On the other hand, if it is calculated in steps (4) and (5), when $h' > h$ , $\alpha_j = \beta_j = 0$ for all Rule-IC, $\beta_j = 0$ for Rule-ICs with $h' > h$, $\beta_j = 1$ for Rule-ICs with $h' < h$, and $\alpha_j = 0$ for Rule-ICs with $h' < h$. Therefore, $u_a(x, \theta) = c' = a / b$ is still the same as $u_a(x, \theta)$ calculated in steps (4) and (5). And similarly, $b' = \sum_{j=1}^{H} \beta_j + \sum_{j=1}^{H} \alpha_j \beta_j$ can be attained. As a result, $c' = c'' = a'/b'$, which is the same as equation (2). By the same token, it can be proved that $c'' = c'' = a'/b'$ will lead to equation (2) when $h' \leq h$ in step (4).

The case of $u_A(x, \theta)$ can be similarly established.

Given the equivalence between the KM algorithm and the EKM algorithm, it should be obvious that all the above theoretical results (i.e., Lemmas 1 to 6 and Theorems 1 and 2) hold for the IT2 Mamdani and TS fuzzy controllers in Section II if they use the EKM algorithm instead of the KM algorithm. For brevity, we will not show proof or state the results formally.

IV. LABORATORY FUZZY CONTROL EXPERIMENT

An IT2 fuzzy controller meeting the requirements set in Section II was designed and implemented for real-time control of a linear voice coil motor (LAL 950-050 model, SMAC Corp.) in our laboratory. The motor consists of a mover and a stator. An electric control signal drives the voice coil in the mover and the resulting force accurately positions the mover. The dynamic input-output relationship of the motor is not linear [36]. Additional nonlinearity exists in the motion actuator, position sensor, A/D converter, and D/A converter. Thus, from the IT2 controller standpoint, the system to be controlled is quite nonlinear. This kind of motors offers fast response and small force ripples, and is widely used in high-speed, high-accuracy positioning control (e.g., precise installation and welding of semiconductor chips). Other components of our system included a motor driver board, an electrical vortex position sensor, a dSPACE DS 1005 host processor, and a Lenovo PC with i7-2600K CPU at 3.40 GHz and 4 GB memory (Fig. 4). The A/D conversion was performed every 5.7 $\mu$s.

For brevity, we will not show proof or state the results formally.

![Block diagram showing fuzzy control experiment's hardware setup.](image)

We use the same notations as in Section II to specify this fuzzy controller: $n = 2$, $[R_c, Q_c] = [-1, 1]$. The fuzzy sets were all triangular and $\mu_c = \frac{1-\theta}{2} \sigma_i + \frac{1+\theta}{2}$, $\mu_c = \frac{1-\theta}{2} \sigma_i + \frac{1+\theta}{2}$, $\mu_c = \frac{1-\theta}{2} \sigma_i + \frac{1+\theta}{2}$ and $\mu_c = \frac{1-\theta}{2} \sigma_i + \frac{1+\theta}{2}$ where $i = 1, 2$. The number of fuzzy rules was 4 (i.e., $M = 4$):
Let's look into the first reason. In control theory, a controller, fuzzy or not, is deemed a variable gain controller if its analytical structure is

\[ u(x) = g_1(x)x_1 + g_2(x)x_2 + \cdots + g_n(x)x_n \]

Here, \( g_1(x), g_2(x), \ldots, g_n(x) \) are controller’s gains. If any of the gains changes with \( x \), that gain is a variable gain, making the controller a nonlinear variable gain controller. On the other hand, if none of the gains varies with \( x \), then all the gain are constant gains. When this is the case, the controller is a linear controller. One popular linear controller is the linear state feedback controller (linear PID controller is a special case of it).

Some typical fuzzy controllers, T1 or IT2, are known to be variable gain controllers (e.g., [33, 35]). Theoretical analyses, real-world applications and computer simulations in the literature reveal that one reason why the T1 controllers can outperform the conventional controllers is their variable gain characteristics (e.g., [29]). Recent research shows that some IT2 controllers are also variable gain controllers with a mechanism manipulating range of the variable gains, which is something impossible to achieve by the T1 controllers [3]. The mechanism is parameterization realized via FOUs of the input fuzzy sets [33, 35]. For the T1 controllers, their gain variation ranges are always fixed. The parameterization provides extra design flexibility, offering the potential for the IT2 controllers to outperform the T1 controllers.

The two two-input IT2 TS controllers whose analytical structures derived in [33, 35] can serve as concrete examples. Both meet the requirements set in Section II. Their analytical structures reveal them to be variable gain controllers. Theorem 2 predicts that the ranges of the variable gains of these controllers become narrower and narrower as the FOUs increase. When the FOUs reach their maximums, the gain variation vanishes and the TS controllers degenerate to piecewise linear controllers. Using the analytical structures provided in [33, 35], correctness of this prediction can be verified, one can draw the same conclusion.

In summary, when the FOUs increase a reduction in variable gain range will take place for at least some of the controllers in Section II. Too large FOUs can lead to too narrow variable gain ranges, potentially worsening control performance. Whether the same can be said to many other controllers is an open interesting research question.

Now let’s look into the second reason for the superiority. How input space is divided into regions and how many of them ranges, potentially worsening control performance. Whether the same can be said to many other controllers is an open interesting research question.

V. DISCUSSION

Numerous real-world applications, laboratory experiments, and computer simulations have empirically established the superiority of an IT2 controller over its T1 counterpart [31]. This is not surprising because the latter is a special case of the former (i.e., FOUs equal 0). An IT2 controller can not only always achieve the exact same control performance obtained by its corresponding T1 controller (by making the FOUs 0), but better performance unattainable by the T1 controller because it has more design parameters (e.g., the FOUs).

Theoretical studies have also been carried out to pinpoint underlying reasons for the observed superiority (e.g., [14, 17, 21]). Among several plausible reasons found in the literature, two are particularly relevant to our findings in Section III. One points to a more powerful variable gain mechanism of the IT2 controllers over their T1 counterparts. The other attributes to finer division of the input space by the IT2 controllers (i.e., a greater sculpting of the state space, as concluded in [14]).

Let’s look into the first reason. In control theory, a controller,
Hence, generally speaking, an IT2 controller with larger FOUs for its input fuzzy sets tends to have fewer regions than the same controller with smaller FOUs. Because one source for good performance of the IT2 controllers is “greater sculpting of the state space” [14], it is logical to conclude that too large FOUs should be avoided.

VI. CONCLUSION

Through investigating the mathematical properties of \( \mu_g(x, \theta) \) and \( \mu_l(x, \theta) \), we prove that the analytical structures of the general IT2 Mamdani and TS controllers monotonically approach constant controllers and piecewise linear controllers, respectively, as the FOUs of their input fuzzy sets increase. The fuzzy controllers become the constant controllers or piecewise linear controllers when all the FOUs are at their maximum. The findings provide an insightful understanding of how the increasing FOUs will affect the analytical structures of the general fuzzy controllers. They are also useful for FOU determination - too large FOUs of the input fuzzy sets are, generally speaking, undesirable because constructing an IT2 controller that will perform similarly to a constant or piecewise linear controller is undesirable.

References


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