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Comparison of Necessary Conditions for Typical Takagi–Sugeno and Mamdani Fuzzy Systems as Universal Approximators

Hao Ying, Yongsheng Ding, Shaokuan Li, and Shihuang Shao

***Abstract*—Both Takagi–Sugeno (TS) and Mamdani fuzzy systems are known to be universal approximators. In this paper, we investigate whether one type of the fuzzy approximators is more economical than the other type. The TS fuzzy systems in this study are the typical two-input single-output TS fuzzy systems: they employ trapezoidal or triangular input fuzzy sets, arbitrary fuzzy rules with linear rule consequent, product fuzzy logic AND, and the centroid defuzzifier. We first establish necessary conditions on minimal system configuration of the TS fuzzy systems as function approximators. We show that the number of the input fuzzy sets and fuzzy rules needed by the TS fuzzy systems depend on the number and locations of the extrema of the function to be approximated. The resulting conditions reveal the strength of the TS fuzzy approximators: only a handful of fuzzy rules may be needed to approximate functions that have complicated formulation but a few extrema. The drawback, though, is that a large number of fuzzy rules must be employed to approximate periodic or highly oscillatory functions. We then compare these necessary conditions with the ones that we established for the general Mamdani fuzzy systems in our previous papers. Results of the comparison unveil that the minimal system configurations of the TS and Mamdani fuzzy systems are comparable. Finally, we prove that the minimal configuration of the TS fuzzy systems can be reduced and becomes smaller than that of the Mamdani fuzzy systems if nontrapezoidal or nontriangular input fuzzy sets are used. We believe that all the results in present paper hold for the TS fuzzy systems with more than two input variables but the proof seems to be mathematically difficult. Our new findings are valuable in designing more compact fuzzy systems, especially fuzzy controllers and models which are two most popular and successful applications of the fuzzy approximators.**

I. INTRODUCTION

Many different configurations of fuzzy systems, either Mamdani type or Takagi–Sugeno (TS) type, have been proved to be universal approximators in that they can uniformly approximate any continuous functions to any degree of accuracy (e.g., [1]–[3], [5], [7]–[12], [14], and [15]). Generally speaking, more fuzzy sets and rules are needed to gain better approximation accuracy [11], [15]. In other words, the smaller the desired error bound is, the more the fuzzy rules are needed. The implication of these results is that fuzzy controllers and models [6], two most popular and successful applications of fuzzy systems, can always produce desired (nonlinear) control and modeling solutions in practice provided that the number of fuzzy sets and rules is allowed to increase as large as necessary.

Practically speaking, one would want to use as simple a fuzzy system (controller or model) as possible to approximate a given function as long as the prespecified accuracy is met. This motivated us

Manuscript received June 12, 1998; revised April 7, 1999. This work was supported in part by a Biomedical Engineering Research Grant from the Whitaker Foundation to the first author and an Advanced Technology Program Grant from the Texas Higher Education Coordinating Board.

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Publisher Item Identifier S 1083-4427(99)06987-8.

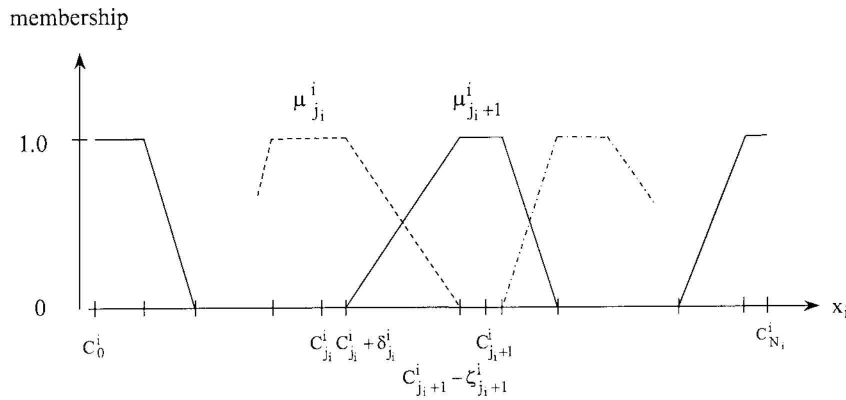

Fig. 1. Graphic description of the membership functions of the input fuzzy sets that are mathematically defined in (1).

to investigate necessary conditions for the Mamdani fuzzy systems as universal approximators with minimal system configuration [4], [13], which exposed the strength as well as limitation of the Mamdani fuzzy systems as approximators. On one hand, only a small number of fuzzy rules may be needed to uniformly approximate multivariate continuous functions that have complicated formulation but a small number of extrema. On the other hand, however, the number of fuzzy rules must be large in order to approximate periodic or highly oscillatory continuous functions.

In this paper, we extend our effort to study TS fuzzy systems on the same aspect. Specifically, we investigate the following two issues that are of theoretical and practical importance.

1) What are the necessary conditions under which typical TS fuzzy systems can possibly be universal approximators but with as minimal system configuration as possible?
2) Given any continuous function, which type of fuzzy systems, TS or Mamdani, is more economical as approximators in that less design parameters are needed?

II. Configuration of Typical TS Fuzzy Systems

The fuzzy systems under this investigation are the typical ones that use input variables x_1 and x_2, where $x_i \in [a_i, b_i]$ and $i = 1, 2$. The interval $[a_i, b_i]$ is divided into N_i subintervals

$$a_i = C_0^i < C_1^i < C_2^i < \cdots < C_{N_i-1}^i < C_{N_i}^i = b_i.$$

On $[a_i, b_i]$, $N_i + 1$ trapezoidal input fuzzy sets, each denoted as $A_{j_i}^i$ $(0 \le j_i \le N_i)$, are defined to fuzzify x_i. $A_{j_i}^i$ has a membership function, designated as $\mu_{j_i}^i(x_i)$, whose mathematical definition is as follows:

$$\mu_{j_i}^i(x_i) = \begin{cases} 0, & x_i \in \left[C_0^i,\, C_{j_i-1}^i + \delta_{j_i-1}^i\right] \\ \varphi_{j_i}^i x_i + \theta_{j_i}^i, & x_i \in \left[C_{j_i-1}^i + \delta_{j_i-1}^i,\, C_{j_i}^i - \zeta_{j_i}^i\right] \\ 1, & x_i \in \left[C_{j_i}^i - \zeta_{j_i}^i,\, C_{j_i}^i + \delta_{j_i}^i\right] \\ \Phi_{j_i}^i x_i + \Theta_{j_i}^i, & x_i \in \left[C_{j_i}^i + \delta_{j_i}^i,\, C_{j_i+1}^i - \zeta_{j_i+1}^i\right] \\ 0, & x_i \in \left[C_{j_i+1}^i - \zeta_{j_i+1}^i,\, C_{N_i}^i\right] \end{cases} \tag{1}$$

where

$$\varphi_{j_i}^i = \frac{1}{\left[C_{j_i}^i - \zeta_{j_i}^i\right] - \left[C_{j_i-1}^i + \delta_{j_i-1}^i\right]},$$

$$\theta_{j_i}^i = -\frac{C_{j_i-1}^i + \delta_{j_i-1}^i}{\left[C_{j_i}^i - \zeta_{j_i}^i\right] - \left[C_{j_i-1}^i + \delta_{j_i-1}^i\right]},$$

$$\Phi_{j_i}^i = -\frac{1}{\left[C_{j_i+1}^i - \zeta_{j_i+1}^i\right] - \left[C_{j_i}^i + \delta_{j_i}^i\right]},$$

$$\Theta_{j_i}^i = \frac{C_{j_i+1}^i - \zeta_{j_i+1}^i}{\left[C_{j_i+1}^i - \zeta_{j_i+1}^i\right] - \left[C_{j_i}^i + \delta_{j_i}^i\right]}.$$

To better understand the definition, we graphically illustrate it in Fig. 1. The membership functions have the following two properties:

1) the trapezoids can be different in upper and lower bases as well as left and right sides;
2) for two neighboring membership functions, say the j_ith and $j_i + 1$th, $\mu_{j_i}^i + \mu_{j_i+1}^i = 1$.

Obviously, the triangular membership functions are just special cases of the trapezoidal ones when $\zeta_{j_i}^i = 0$ and $\delta_{j_i}^i = 0$. In this paper, we call each combination of $[C_{j_1}^1, C_{j_1+1}^1] \times [C_{j_2}^2, C_{j_2+1}^2]$ a cell on $[a_1, b_1] \times [a_2, b_2]$.

The TS fuzzy systems use arbitrary fuzzy rules with linear rule consequent

$$\begin{aligned} &\text{IF } x_1 \text{ is } A_{h_1}^1 \text{ AND } x_2 \text{ is } A_{h_2}^2 \text{ THEN } F(x_1, x_2) \\ &\quad = \alpha_{h_1, h_2} x_1 + \beta_{h_1, h_2} x_2 + \gamma_{h_1, h_2} \end{aligned} \tag{2}$$

where α_{h_1, h_2}, β_{h_1, h_2}, and γ_{h_1, h_2} can be any constants chosen by the system developer and $F(x_1, x_2)$ designates output of the fuzzy systems. Product fuzzy logic AND is employed to yield combined membership $\mu_{h_1}^1 \mu_{h_2}^2$ for the rule consequent. Using the popular centroid defuzzifier and noting $\mu_{j_i}^i + \mu_{j_i+1}^i = 1$, we obtain

$$\begin{aligned} F(x_1, x_2) &= \frac{\sum \mu_{h_1}^1 \mu_{h_2}^2 (\alpha_{h_1, h_2} x_1 + \beta_{h_1, h_2} x_2 + \gamma_{h_1, h_2})}{\sum \mu_{h_1}^1 \mu_{h_2}^2} \\ &= \sum \mu_{h_1}^1 \mu_{h_2}^2 (\alpha_{h_1, h_2} x_1 + \beta_{h_1, h_2} x_2 + \gamma_{h_1, h_2}). \end{aligned}$$

We point out that the configuration of the TS fuzzy systems described above is typical and commonly used in fuzzy control and modeling. Moreover, we have proved in our previous paper that these TS fuzzy systems are universal approximators and have also derived a formula for computing the needed number of input fuzzy sets and rules based on the function to be approximated as well as prespecified approximation accuracy [12].

III. Necessary Conditions on Minimal System Configuration for the Typical TS Fuzzy Systems as Universal Approximators

In this section, we will establish necessary conditions on minimal system configuration requirement for the typical TS fuzzy systems as function approximators. We assume the following information is available:

1) an arbitrarily small approximation error bound $\varepsilon > 0$;
2) the continuous function to be approximated, designated as $f(x_1, x_2)$, has K distinctive extrema on $(a_1, b_1) \times (a_2, b_2)$.

These two assumptions are minimum and very nonrestrictive, and can indeed be obtained in practice if the function to be approximated is readily measurable.

For a more clear and concise presentation of the mathematical proof of the conditions, we first need to establish the following lemmas.

Lemma 1: $F(x_1, x_2)$ is continuous on $[a_1, b_1] \times [a_2, b_2]$ if and only if the following two conditions are met:

1) Four different fuzzy rules in the form of (2) are assigned to each of the $N_1 \times N_2$ combinations of subintervals.
2) $(N_1 + 1)(N_2 + 1)$ fuzzy rules are used for the $N_1 \times N_2$ combinations of subintervals.

Proof: The proof is similar to that we gave to the general Mamdani fuzzy systems [4], [13]. Basically, for each cell $[C_{j_1}^1, C_{j_1+1}^1] \times [C_{j_2}^2, C_{j_2+1}^2]$, two nonzero memberships are resulted for x_1 and another two for x_2 after fuzzification. Hence, there are four different combinations of the four memberships, leading to activation of four fuzzy rules. Four rules must be used in order to gain continuity of $F(x_1, x_2)$ on $[C_{j_1}^1, C_{j_1+1}^1] \times [C_{j_2}^2, C_{j_2+1}^2]$.

Furthermore, there exist a total of $(N_1 + 1)(N_2 + 1)$ different membership combinations, resulting in the need of the same number of fuzzy rules if continuity of $F(x_1, x_2)$ on $[a_1, b_1] \times [a_2, b_2]$ is wanted. ■

Lemma 2: The following third-order function

$$P(x_1, x_2) = a + bx_1 + cx_2 + dx_1x_2 + ex_1^2 + fx_2^2 + gx_1^2x_2 + hx_1x_2^2,$$

where a, b, c, d, e, f, g, and h can be any real constants and $x_1, x_2 \in (-\infty, \infty)$, has at most one extremum.

Proof: We assume that (x_1^*, x_2^*) is one extreme point of $P(x_1, x_2)$ and prove that there exists at most one extreme point on entire $(-\infty, \infty) \times (-\infty, \infty)$. We first shift the origin of the $x_1 - x_2$ coordinate system from (0, 0) to (x_1^*, x_2^*) by letting $x_1 = \overline{x}_1 + x_1^*$, $x_2 = \overline{x}_2 + x_2^*$, resulting in a new third-order function

$$\begin{aligned} G(\overline{x}_1, \overline{x}_2) &= P(\overline{x}_1 + x_1^*, \overline{x}_2 + x_2^*) \\ &= a + b(\overline{x}_1 + x_1^*) + c(\overline{x}_2 + x_2^*) + d(\overline{x}_1 + x_1^*)(\overline{x}_2 + x_2^*) \\ &\quad + e(\overline{x}_1 + x_1^*)^2 + f(\overline{x}_2 + x_2^*)^2 + g(\overline{x}_1 + x_1^*)^2(\overline{x}_2 + x_2^*) \\ &\quad + h(\overline{x}_1 + x_1^*)(\overline{x}_2 + x_2^*)^2 \\ &= \overline{a} + \overline{b}\,\overline{x}_1 + \overline{c}\,\overline{x}_2 + \overline{d}\,\overline{x}_1\overline{x}_2 + \overline{e}\,\overline{x}_1^2 + \overline{f}\,\overline{x}_2^2 + \overline{g}\,\overline{x}_1^2\overline{x}_2 + \overline{h}\,\overline{x}_1\overline{x}_2^2 \end{aligned}$$

where $\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}, \overline{g}$, and $\overline{h}$ are any constants (they are computed from $a, b, c, d, e, f, g, h, x_1^*$, and x_2^*). We now look for all possible extreme points of $G(\overline{x}_1, \overline{x}_2)$ by doing the following:

$$\frac{\partial G}{\partial \overline{x}_1} = \overline{b} + \overline{d}\,\overline{x}_2 + 2\overline{e}\,\overline{x}_1 + 2\overline{g}\,\overline{x}_1\overline{x}_2 + \overline{h}\,\overline{x}_2^2, \quad (3)$$

$$\frac{\partial G}{\partial \overline{x}_2} = \overline{c} + \overline{d}\,\overline{x}_1 + 2\overline{f}\,\overline{x}_2 + \overline{g}\,\overline{x}_1^2 + 2\overline{h}\,\overline{x}_1\overline{x}_2, \quad (4)$$

$$\frac{\partial^2 G}{\partial \overline{x}_1\,\partial \overline{x}_2} = \frac{\partial^2 G}{\partial \overline{x}_2\,\partial \overline{x}_1} = \overline{d} + 2\overline{g}\,\overline{x}_1 + 2\overline{h}\,\overline{x}_2,$$

$$\frac{\partial^2 G}{\partial \overline{x}_1^2} = 2\overline{e} + 2\overline{g}\,\overline{x}_2, \qquad \frac{\partial^2 G}{\partial \overline{x}_2^2} = 2\overline{f} + 2\overline{h}\,\overline{x}_1,$$

$$\begin{aligned} D &= \begin{vmatrix} \dfrac{\partial^2 G}{\partial \overline{x}_1^2} & \dfrac{\partial^2 G}{\partial \overline{x}_1\,\partial \overline{x}_2} \\ \dfrac{\partial^2 G}{\partial \overline{x}_2\,\partial \overline{x}_1} & \dfrac{\partial^2 G}{\partial \overline{x}_2^2} \end{vmatrix} \\ &= 4(\overline{e} + \overline{g}\,\overline{x}_2)(\overline{f} + \overline{h}\,\overline{x}_1) - (\overline{d} + 2\overline{g}\,\overline{x}_1 + 2\overline{h}\,\overline{x}_2)^2. \end{aligned} \quad (5)$$

The sufficient condition for $(\overline{x}_1, \overline{x}_2) = (0, 0)$ [equivalently, $(x_1, x_2) = (x_1^*, x_2^*)$] to be an extreme point is

$$D = 4\overline{e}\,\overline{f} - \overline{d}^2 > 0, \quad (6)$$

which means if we properly choose the values of $\overline{e}$, $\overline{f}$, and $\overline{d}$ so that the condition $D > 0$ is satisfied, then our assumption of (0, 0) being an extreme point indeed holds.

Now we show that except (0, 0), there exist no other extreme points. Because (0, 0) can be an extreme point, we have

$$\left.\frac{\partial G}{\partial \overline{x}_1}\right|_{(0,\,0)} = 0 \quad \text{and} \quad \left.\frac{\partial G}{\partial \overline{x}_2}\right|_{(0,\,0)} = 0$$

and hence $\overline{b} = 0$ in (3) and $\overline{c} = 0$ in (4). As a result, any other possible extreme points must be the solutions of the following equation set:

$$\begin{cases} \overline{d}\,\overline{x}_2 + 2\overline{e}\,\overline{x}_1 + 2\overline{g}\,\overline{x}_1\overline{x}_2 + \overline{h}\,\overline{x}_2^2 = 0 \\ \overline{d}\,\overline{x}_1 + 2\overline{f}\,\overline{x}_2 + \overline{g}\,\overline{x}_1^2 + 2\overline{h}\,\overline{x}_1\overline{x}_2 = 0 \end{cases}$$

which can be written either as

$$\begin{cases} 2\overline{e}\,\overline{x}_1 + (\overline{d} + 2\overline{g}\,\overline{x}_1 + \overline{h}\,\overline{x}_2)\overline{x}_2 = 0 \\ (\overline{d} + \overline{g}\,\overline{x}_1 + 2\overline{h}\,\overline{x}_2)\overline{x}_1 + 2\overline{f}\,\overline{x}_2 = 0 \end{cases} \quad (7)$$

or

$$\begin{cases} (2\overline{e} + 2\overline{g}\,\overline{x}_2)\overline{x}_1 + (\overline{d} + \overline{h}\,\overline{x}_2)\overline{x}_2 = 0 \\ (\overline{d} + \overline{g}\,\overline{x}_1)\overline{x}_1 + (2\overline{f} + 2\overline{h}\,\overline{x}_1)\overline{x}_2 = 0. \end{cases} \quad (8)$$

The necessary and sufficient conditions for (7) and (8) to have nonzero solutions are, respectively,

$$\begin{vmatrix} 2\overline{e} & \overline{d} + \overline{h}\,\overline{x}_2 + 2\overline{g}\,\overline{x}_1 \\ \overline{d} + \overline{g}\,\overline{x}_1 + 2\overline{h}\,\overline{x}_2 & 2\overline{f} \end{vmatrix} = 0 \quad (9)$$

and

$$\begin{vmatrix} 2\overline{e} + 2\overline{g}\,\overline{x}_2 & \overline{d} + \overline{h}\,\overline{x}_2 \\ \overline{d} + \overline{g}\,\overline{x}_1 & 2\overline{f} + 2\overline{h}\,\overline{x}_1 \end{vmatrix} = 0. \quad (10)$$

After some simple derivation, we obtain from (9)

$$\begin{aligned} &(\overline{d} + 2\overline{h}\,\overline{x}_2 + 2\overline{g}\,\overline{x}_1)^2 \\ &\quad = (\overline{g}\,\overline{x}_1 + \overline{h}\,\overline{x}_2)(\overline{d} + 2\overline{h}\,\overline{x}_2 + 2\overline{g}\,\overline{x}_1) + 4\overline{e}\,\overline{f} - \overline{g}\,\overline{h}\overline{x}_1\overline{x}_2 \end{aligned} \quad (11)$$

and from (10), we obtain

$$4(\overline{e} + \overline{g}\,\overline{x}_2)(\overline{f} + \overline{h}\,\overline{x}_1) = (\overline{d} + \overline{g}\,\overline{x}_1)(\overline{d} + \overline{h}\,\overline{x}_2). \quad (12)$$

Replacing the two terms in (5) by (11) and (12), respectively, we gain

$$\begin{aligned} D &= (\overline{d} + \overline{g}\,\overline{x}_1)(\overline{d} + \overline{h}\,\overline{x}_2) - (\overline{g}\,\overline{x}_1 + \overline{h}\,\overline{x}_2)(\overline{d} + 2\overline{h}\,\overline{x}_2 + 2\overline{g}\,\overline{x}_1) \\ &\quad - 4\overline{e}\,\overline{f} + \overline{g}\,\overline{h}\,\overline{x}_1\overline{x}_2 \\ &= -2\big((\overline{h}\,\overline{x}_2)^2 + (\overline{g}\,\overline{x}_1)^2 - \overline{h}\,\overline{x}_2\overline{g}\,\overline{x}_1\big) - (4\overline{e}\,\overline{f} - \overline{d}^2) < 0. \end{aligned}$$

In the last step, we used the well-known inequality

$$(\overline{h}\,\overline{x}_2)^2 + (\overline{g}\,\overline{x}_1)^2 - \overline{h}\,\overline{x}_2\overline{g}\,\overline{x}_1 \geq 0$$

and inequality (6).

The fact that $D < 0$ on entire $(-\infty, \infty) \times (-\infty, \infty)$, excluding (0, 0), means there does not exist any other extreme points except (0, 0). This completes our proof. ■

Lemma 3: The following second-order functions

$$P(x_1, x_2) = a + bx_1 + cx_2 + dx_1x_2 + ex_1^2$$
$$Q(x_1, x_2) = a + bx_1 + cx_2 + dx_1x_2 + ex_2^2$$

where a–e can be any real constants, are monotonic on entire $(-\infty, \infty) \times (-\infty, \infty)$.

Proof: The proof is straightforward as the following derivations show:

$$\frac{\partial P}{\partial x_1} = b + dx_2 + 2ex_1, \qquad \frac{\partial P}{\partial x_2} = c + dx_1,$$

$$\frac{\partial^2 P}{\partial x_1^2} = 2e, \qquad \frac{\partial^2 P}{\partial x_1\,\partial x_2} = \frac{\partial^2 P}{\partial x_2\,\partial x_1} = d, \qquad \frac{\partial^2 P}{\partial x_2^2} = 0,$$

$$D = \begin{vmatrix} \dfrac{\partial^2 P}{\partial x_1^2} & \dfrac{\partial^2 P}{\partial x_1\,\partial x_2} \\ \dfrac{\partial^2 P}{\partial x_2\,\partial x_1} & \dfrac{\partial^2 P}{\partial x_2^2} \end{vmatrix} = -d^2.$$

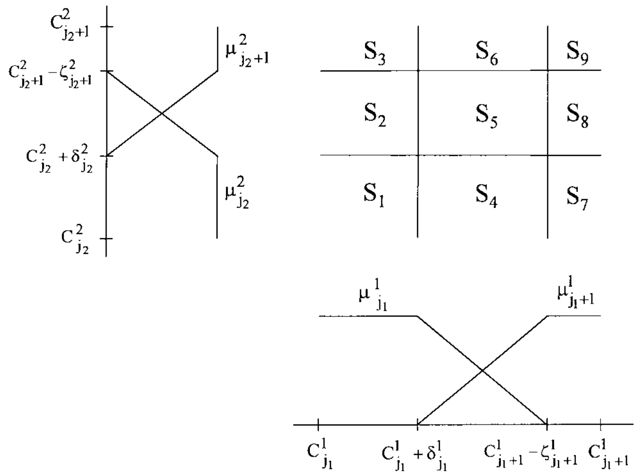

Fig. 2. Division of the input space into nine regions for proving that the typical TS fuzzy systems have at most one extremum in the whole input space.

$P(x_1, x_2)$ does not have any extreme on entire $(-\infty, \infty) \times (-\infty, \infty)$ because $D = -d^2 \leq 0$. This conclusion obviously also holds for $Q(x_1, x_2)$. ■

Having established these three lemmas, we are now ready to prove the following main results.

Theorem 1: When Lemma 1 holds, the TS fuzzy systems have at most one extremum in each of the $N_1 \times N_2$ combinations of subintervals.

Proof: Without losing generality, assume $x_1 \in [C^1_{j_1}, C^1_{j_1+1}]$ and $x_2 \in [C^2_{j_2}, C^2_{j_2+1}]$. After fuzzification, only two nonzero memberships are resulted for each input variable and they are

$$\mu^1_{j_1} \quad \text{and} \quad \mu^1_{j_1+1} \qquad \text{for } x_1$$
$$\mu^2_{j_2} \quad \text{and} \quad \mu^2_{j_2+1} \qquad \text{for } x_2.$$

Consequently, four rules relating to these memberships are activated. To investigate how many extrema exist, we need to divide $[C^1_{j_1}, C^1_{j_1+1}] \times [C^2_{j_2}, C^2_{j_2+1}]$ into nine regions, as shown in Fig. 2. In region S_5, the output of the TS fuzzy systems is a third-order function

$$\begin{aligned} F(x_1, x_2) &= \sum \mu^1_{h_1}\mu^2_{h_2}(\alpha_{h_1,h_2}x_1 + \beta_{h_1,h_2}x_2 + \gamma_{h_1,h_2}) \\ &= p_0 + p_1x_1 + p_2x_2 + p_3x_1x_2 + p_4x_1^2 + p_5x_2^2 \\ &\quad + p_6x_1^2x_2 + p_7x_1x_2^2 \end{aligned} \tag{13}$$

where coefficients p_0–p_7 are constants whose values are determined by the membership functions of the input fuzzy sets as well as by the parameters in the fuzzy rule consequent. In regions S_4 and S_6

$$F(x_1, x_2) = p_0 + p_1x_1 + p_2x_2 + p_3x_1x_2 + p_4x_1^2 \tag{14}$$

whereas in regions S_2 and S_8

$$F(x_1, x_2) = p_0 + p_1x_1 + p_2x_2 + p_3x_1x_2 + p_4x_2^2, \tag{15}$$

both of which are second-order functions. Finally, in regions S_1, S_3, S_7, and S_9, the fuzzy systems output is in the form of plane (i.e., first-order function):

$$F(x_1, x_2) = p_0 + p_1x_1 + p_2x_2. \tag{16}$$

In different regions, the values of coefficients p_0–p_7 in (13)–(16) are different. As example, we provide the explicit expressions of the coefficients for regions S_1, S_2, and S_5 in the Appendix.

According to Lemma 2, $F(x_1, x_2)$ has at most one extremum in region S_5. Due to Lemma 3, $F(x_1, x_2)$ is monotonic in regions S_2, S_4, S_6, and S_8. Being planes, $F(x_1, x_2)$ is also monotonic in regions S_1, S_3, S_7, and S_9. Note that $F(x_1, x_2)$ is continuous on $[C^1_{j_1}, C^1_{j_1+1}] \times [C^2_{j_2}, C^2_{j_2+1}]$, when Lemma 1 holds. Therefore, $F(x_1, x_2)$ has at most one extremum on $[C^1_{j_1}, C^1_{j_1+1}] \times [C^2_{j_2}, C^2_{j_2+1}]$. ■

Recall that in the beginning of this section, we assumed the continuous function to be approximated, $f(x_1, x_2)$, has K distinctive extrema at (x_1^j, x_2^j), where $j = 1, 2, \cdots, K$, on $(a_1, b_1) \times (a_2, b_2)$. We now prove, using Lemmas 1–3 and Theorem 1, the necessary conditions for the TS fuzzy systems as universal function approximators with minimal system configuration.

Theorem 2: To approximate $f(x_1, x_2)$ with arbitrarily small error bound, one must choose such N_1 and N_2 that, respectively, divide $[a_1, b_1]$ and $[a_2, b_2]$ in such a way that at most one extremum exists in each cell $[C^1_{j_1}, C^1_{j_1+1}] \times [C^2_{j_2}, C^2_{j_2+1}]$ for the typical TS fuzzy systems. Accordingly, the minimal number of fuzzy rules needed is $(N_1+1)(N_2+1)$, with $3(N_1+1)(N_2+1)$ parameters in the rule consequent.

Proof: In order to approximate $f(x_1, x_2)$ arbitrarily well, one must first approximate all the extrema arbitrarily well, which means that the output of the TS fuzzy systems must reach the extrema at (x_1^j, x_2^j) for all j. According to Theorem 1, the TS fuzzy systems have at most one extremum in each cell, regardless of the size of the cell. Hence, one must divide $[a_1, b_1]$ and $[a_2, b_2]$ in such a way that at most one extremum of $f(x_1, x_2)$ exists in each cell $[C^1_{j_1}, C^1_{j_1+1}] \times [C^2_{j_2}, C^2_{j_2+1}]$, $j_1 = 1, 2, \cdots, N_1$, and $j_2 = 1, 2, \cdots, N_2$. Based on Lemma 1, the TS fuzzy systems need $(N_1+1)(N_2+1)$ fuzzy rules. Since there are three parameters in each rule consequent [see (2)], total $3(N_1+1)(N_2+1)$ parameters are required in all the rule consequent.

In many cases, additional rules are needed to approximate whole $f(x_1, x_2)$, not just the extrema, as accurately as desired. ■

Theorem 2 shows that, as universal approximators, the TS fuzzy systems have similar strength and limitation possessed by the general Mamdani fuzzy systems that we studied before [4], [13]. On one hand, it is possible for the TS fuzzy systems to use only a handful of fuzzy rules to uniformly and accurately approximate functions that are complicated but only have a few extrema. This explains why the majority of successful TS fuzzy controllers and models in the literature need to employ only a small number of fuzzy rules to achieve satisfactory results. On the other hand, a large amount of fuzzy rules is required for approximating simple functions with many extrema. The number of fuzzy rules needed increases with the increase of the number of extrema of $f(x_1, x_2)$. This means that the fuzzy systems are not ideal function approximators for periodic or highly-oscillatory functions.

So far in present paper, we have studied minimal system configuration of the TS fuzzy systems as function approximators purely from mathematical standpoint. There have already existed many different function approximators, such as polynomial and spline functions, in traditional function approximation theory. As always, each type of approximators has its advantage and limitation. The distinctive advantage of fuzzy approximators over other approximators lies in their unique ability to utilizing not only numerical data but also linguistically-expressed human knowledge and experience.

IV. Minimal System Configuration Comparison Between the TS and Mamdani Fuzzy Systems as Universal Approximators

In our previous papers [4], [13], we established the necessary conditions on minimal system configuration for the general MISO Mamdani fuzzy systems as universal approximators. These Mamdani fuzzy systems employ almost arbitrary continuous input fuzzy sets, arbitrary singleton output fuzzy sets, arbitrary fuzzy rules, product fuzzy logic AND and the generalized defuzzifier containing the

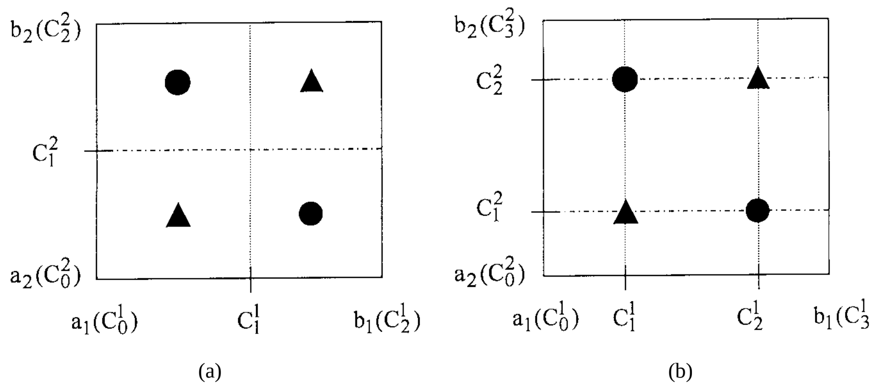

Fig. 3. Comparison of the minimal system configurations of the typical TS fuzzy systems and the general Mamdani fuzzy systems. The example function to be approximated has two maximum points, whose locations are marked by symbol ▲, and two minimum points whose locations are marked by symbol •. (a) gives one possible division of the input space for the TS fuzzy systems to be minimal whereas (b) provides the necessary input space division for the Mamdani fuzzy systems to be minimal.

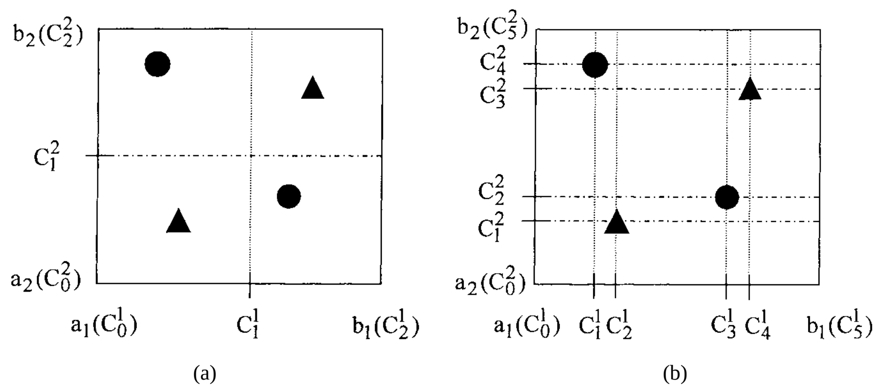

Fig. 4. Comparison of the minimal system configurations of the typical TS fuzzy systems and the general Mamdani fuzzy systems using another example function. The meanings of the symbols are the same as those in Fig. 3. This example function has the same number of extrema but the locations of the minimum points are slightly different from those displayed in Fig. 3. (a) gives one possible division of the input space for the TS fuzzy systems to be minimal whereas (b) provides the necessary input space division for the Mamdani fuzzy systems to be minimal.

centroid defuzzifier as a special case. The conditions are virtually the same as those established in present paper. In this section, we will compare the necessary conditions developed above for the TS fuzzy systems with those we previously established for the general Mamdani fuzzy systems. The purpose of the comparison is to determine whether one type of the fuzzy systems is more economical than the other types. We will use the following two simple yet representative examples to show our points and reach our conclusions for the comparison.

In the first example, the function to be approximated has two maximum points whose locations are marked by symbol ▲, and two minimum points whose locations are marked by symbol •, as shown in Fig. 3(a) [we use the same symbols in Figs. 3(b) and 4(a) and (b)]. According to Theorem 2, $N_1 = N_2 = 2$ and we give, as shown in Fig. 3(a), one possible way to divide both $[a_1, b_1]$ and $[a_2, b_2]$ into two intervals. Correspondingly, at least nine fuzzy rules with 27 rule consequent parameters are needed by the TS fuzzy systems. The system developer will have to determine 27 parameters. However, for the same function, we must divide both $[a_1, b_1]$ and $[a_2, b_2]$ into three intervals, as shown in Fig. 3(b), according to Theorem 2 in our previous paper [4] (also see [13]). Hence, only 16 fuzzy rules are required by the Mamdani fuzzy systems. This means only 16 parameters, each is a singleton output fuzzy sets, need to be determined by the developer. Obviously, the TS fuzzy systems are less economical than the Mamdani fuzzy systems because of the larger number of design parameters.

Fig. 4(a) shows our second example function to be approximated that also has two maximum points and two minimum points. The locations of the two minimum points are slightly different from those in Fig. 3. In this case, the division of $[a_1, b_1]$ and $[a_2, b_2]$ can be the same as that in Fig. 3(a) and the minimal configuration requirement for the TS fuzzy systems remains the same, that is, 27 parameters. Nevertheless, the optimal division of $[a_1, b_1]$ and $[a_2, b_2]$ for the Mamdani fuzzy systems now must be that shown in Fig. 4(b) where $N_1 = N_2 = 5$. The corresponding number of fuzzy rules is 36. Hence, the minimal system configuration of the TS fuzzy systems is more economical.

Through these two examples, one sees that the minimal system configuration of the TS and Mamdani fuzzy systems depends on how many extrema the function to be approximated has and where they are. For some functions, the TS fuzzy systems are more economical whereas for others the Mamdani fuzzy systems are smaller in the number of design parameters. For all the functions as whole, these two types of fuzzy systems are comparably economical and no one is better or worse than the other.

In the minimal system configuration comparison thus far, we have limited the input fuzzy sets of the TS fuzzy systems to trapezoidal/triangular types, as defined in Section II. Would the comparison outcome be different if nontrapezoidal/nontriangular input fuzzy sets are used? Our answer is yes. In what follows, we show that as far as minimal system configuration is concerned, it is advantageous for the TS fuzzy systems to use nontrapezoidal/nontriangular input fuzzy sets. This is because they can make the TS fuzzy systems have more than one extremum in each cell, subsequently reducing the number of fuzzy rules needed. A general mathematical proof of this new finding is difficult because there exist countless different types of nontrapezoidal/nontriangular fuzzy sets and explicitly describing all of them is impossible. Alternatively, we rigorously prove our finding using some typical single-input single-output (SISO) TS fuzzy systems. For notional consistence, we will use x_1 only along with all the other associated notations created in Section II to describe the configuration of these SISO TS fuzzy systems.

For the SISO fuzzy systems involved, membership functions of the input fuzzy sets are defined as

$$\mu_{j_1}^1(x_1)=\begin{cases}0, & x_1\in\left[C_0^1,\, C_{j_1-1}^1+\delta_{j_1-1}^1\right]\\ I_{j_1}^1(x_1), & x_1\in\left[C_{j_1-1}^1+\delta_{j_1-1}^1,\, C_{j_1}^1-\zeta_{j_1}^1\right]\\ 1, & x_1\in\left[C_{j_1}^1-\zeta_{j_1}^1,\, C_{j_1}^1+\delta_{j_1}^1\right]\\ D_{j_1}^1(x_1), & x_1\in\left[C_{j_1}^1+\delta_{j_1}^1,\, C_{j_1+1}^1-\zeta_{j_1+1}^1\right]\\ 0, & x_1\in\left[C_{j_1+1}^1-\zeta_{j_1+1}^1,\, C_{N_1}^1\right]\end{cases}$$

where $I_{j_1}^1(x_1)$ is a monotonically increasing function whereas $D_{j_1}^1(x_1)$ is a monotonically decreasing function. Their values are within [0, 1]. This definition is the same as that in (1) except the two linear functions in (1) are replaced by $I_{j_1}^1(x_1)$ and $D_{j_1}^1(x_1)$. We specifically choose $\delta_{j_1}^1=\zeta_{j_1}^1=0$ for $j_1=1,\,2,\,\cdots,\,N_1$ and let

$$I_{j_1}^1(x_1)=1-\left(\frac{x_1-C_{j_1-1}^1}{C_{j_1}^1-C_{j_1-1}^1}\right)^2$$

and

$$D_{j_1}^1(x_1)=\left(\frac{x_1-C_{j_1}^1}{C_{j_1+1}^1-C_{j_1}^1}\right)^2.$$

After defuzzification, the output of the typical SISO fuzzy systems on $[C_{j_1}^1,\, C_{j_1+1}^1]$ is

$$F(x_1)=\frac{\mu_{j_1}^1(x_1)(\alpha_{j_1}^1x_1+\gamma_{j_1}^1)+\mu_{j_1+1}^1(x_1)(\alpha_{j_1+1}^1x_1+\gamma_{j_1+1}^1)}{\mu_{j_1}^1(x_1)+\mu_{j_1+1}^1(x_1)} \tag{17}$$

and we let the rule consequent parameters be: $\alpha_{j_1}^1=\frac{1}{4}$, $\gamma_{j_1}^1=0$, $\alpha_{j_1+1}^1=\frac{5}{4}$, and $\gamma_{j_1+1}^1=-1$. Without losing generality, we suppose $C_{j_1}^1=0$ and $C_{j_1+1}^1=1$. Using the specific definition of $\mu_{j_1}^1(x_1)$ and the rule consequent parameters in (17), we obtain

$$F(x_1)=\tfrac{1}{4}x_1-x_1^2+x_1^3, \qquad x_1\in[0,\,1].$$

It is easy to prove that $F(x_1)$ reaches maximum at $x_1=\frac{1}{6}$ and minimum at $x_1=\frac{1}{2}$. This means there are two extrema in [0, 1]. These typical SISO TS fuzzy systems can have more than one extremum in a cell because the membership functions are no long limited to trapezoidal or triangular shapes. When multiple extrema exist in some cells, the number of subintervals on $[a_1,\, b_1]$ can be small even when the number of extrema of the function to be approximated is large. Hence, the SISO TS fuzzy systems can be more economic in minimal system configuration than the general SISO Mamdani fuzzy systems because the output of latter is always monotonic in a cell, regardless of the shape of the membership functions [4], [13].

The conclusions drawn from the above analysis of the SISO TS and Mamdani fuzzy systems also hold for the TS and the general Mamdani fuzzy systems with two input variables. We now conclude this section by summarizing all the above comparison results regarding minimal system configurations in the form of following theorem:

Theorem 3: The minimal configurations of the typical TS fuzzy systems and the general Mamdani fuzzy systems depend on the number and location of the extrema of the function to be approximated. When trapezoidal/triangular input fuzzy sets are used, the TS and Mamdani fuzzy systems are comparable in minimal system configuration. Use of nontrapezoidal/nontriangular input fuzzy sets can minimize configuration for the typical TS fuzzy systems, resulting in smaller configuration as compared to the general Mamdani fuzzy systems.

V. Conclusions

We have established necessary conditions for the typical TS fuzzy systems as function approximators with as small a system configuration as possible. We have proved that the number of input fuzzy sets used by the TS fuzzy systems depend on the number and locations of extrema of the function to be approximated. We have compared these conditions with the ones that we previously established for the general Mamdani fuzzy systems. Results of the comparison reveal that, when trapezoidal or triangular input fuzzy sets are used, the typical TS fuzzy systems and the general Mamdani fuzzy systems have comparable minimal system configuration. Furthermore, we have found that the TS fuzzy systems can be more economical in the number of input fuzzy sets and fuzzy rules than the general Mamdani fuzzy systems if nontrapezoidal/nontriangular input fuzzy sets are used. Our new findings are valuable in designing more compact fuzzy systems, such as fuzzy controllers and models which are two most popular and successful applications of the fuzzy approximators.

We believe that all the results in present paper hold for the TS fuzzy systems with more than two input variables. A rigorous proof seems to be mathematically challenging and is an interesting and valuable research topic.

Appendix

To carry out the proof in Theorem 1, we need the explicit expressions of coefficients p_0–p_7 for all the nine regions shown in Fig. 2. For brevity, we give here, without showing the detail derivations, the expressions for coefficients p_0–p_7 when input variables of the TS fuzzy systems are in regions S_1, S_2, and S_5.

For region S_1, $x_1\in[C_{j_1}^1,\, C_{j_1}^1+\delta_{j_1}^1]$, and $x_2\in[C_{j_2}^2,\, C_{j_2}^2+\delta_{j_2}^2]$, output of the TS fuzzy systems is

$$F(x_1,\,x_2)=p_0+p_1x_1+p_2x_2,$$

where

$$p_0=\gamma_{j_1,\,j_2},\qquad p_1=\alpha_{j_1,\,j_2},\quad \text{and}\quad p_2=\beta_{j_1,\,j_2}.$$

For region S_2, $x_1\in[C_{j_1}^1,\, C_{j_1}^1+\delta_{j_1}^1]$ and $x_2\in[C_{j_2}^2+\delta_{j_2}^2,\, C_{j_2+1}^2-\zeta_{j_2+1}^2]$, output of the TS fuzzy systems is

$$F(x_1,\,x_2)=p_0+p_1x_1+p_2x_2+p_3x_1x_2+p_4x_2^2,$$

where

$$\begin{aligned}p_0&=\Theta_{j_2}^2\gamma_{j_1,\,j_2}+\Theta_{j_2+1}^2\gamma_{j_1,\,j_2+1},\\ p_1&=\Theta_{j_2}^2\alpha_{j_1,\,j_2}+\Theta_{j_2+1}^2\alpha_{j_1,\,j_2+1},\\ p_2&=\Phi_{j_2}^2\gamma_{j_1,\,j_2}+\Theta_{j_2}^2\beta_{j_1,\,j_2}+\Phi_{j_2+1}^2\gamma_{j_1,\,j_2+1}\\ &\quad+\Theta_{j_2+1}^2\beta_{j_1,\,j_2+1},\\ p_3&=\Phi_{j_2}^2\alpha_{j_1,\,j_2}+\Phi_{j_2+1}^2\alpha_{j_1,\,j_2+1},\\ p_4&=\Phi_{j_2}^2\beta_{j_1,\,j_2}+\Phi_{j_2+1}^2\beta_{j_1,\,j_2+1}.\end{aligned}$$

For region S_5, $x_1 \in [C^1_{j_1} + \delta^1_{j_1}, C^1_{j_1+1} - \zeta^1_{j_1+1}]$ and $x_2 \in [C^2_{j_2} + \delta^2_{j_2}, C^2_{j_2+1} - \zeta^2_{j_2+1}]$, output of the TS fuzzy systems is:

$$\begin{aligned}F(x_1, x_2) = {} & p_0 + p_1 x_1 + p_2 x_2 + p_3 x_1 x_2 + p_4 x_1^2 + p_5 x_2^2 \\ & + p_6 x_1^2 x_2 + p_7 x_1 x_2^2,\end{aligned}$$

where

$$\begin{aligned}
p_0 = {} & \theta^1_{j_1}\Theta^2_{j_2}\gamma_{j_1, j_2} + \theta^1_{j_1}\Theta^2_{j_2+1}\gamma_{j_1, j_2+1} + \theta^1_{j_1+1}\Theta^2_{j_2}\gamma_{j_1+1, j_2} \\
& + \theta^1_{j_1+1}\Theta^2_{j_2+1}\gamma_{j_1+1, j_2+1}, \\
p_1 = {} & \varphi^1_{j_1}\Theta^2_{j_2}\gamma_{j_1, j_2} + \theta^1_{j_1}\Theta^2_{j_2}\alpha_{j_1, j_2} + \varphi^1_{j_1}\Theta^2_{j_2+1}\gamma_{j_1, j_2+1} \\
& + \theta^1_{j_1}\Theta^2_{j_2+1}\alpha_{j_1, j_2+1} + \varphi^1_{j_1+1}\Theta^2_{j_2}\gamma_{j_1+1, j_2} \\
& + \theta^1_{j_1+1}\Theta^2_{j_2}\alpha_{j_1+1, j_2} + \varphi^1_{j_1+1}\Theta^2_{j_2+1}\gamma_{j_1+1, j_2+1} \\
& + \theta^1_{j_1+1}\Theta^2_{j_2+1}\alpha_{j_1+1, j_2+1}, \\
p_2 = {} & \theta^1_{j_1}\Phi^2_{j_2}\gamma_{j_1, j_2} + \theta^1_{j_1}\Theta^2_{j_2}\beta_{j_1, j_2} + \theta^1_{j_1}\Phi^2_{j_2+1}\gamma_{j_1, j_2+1} \\
& + \theta^1_{j_1}\Theta^2_{j_2+1}\beta_{j_1, j_2+1} + \theta^1_{j_1+1}\Phi^2_{j_2}\gamma_{j_1+1, j_2} \\
& + \theta^1_{j_1+1}\Theta^2_{j_2}\beta_{j_1+1, j_2} + \theta^1_{j_1+1}\Phi^2_{j_2+1}\gamma_{j_1+1, j_2+1} \\
& + \theta^1_{j_1+1}\Theta^2_{j_2+1}\beta_{j_1+1, j_2+1}, \\
p_3 = {} & \varphi^1_{j_1}\Phi^2_{j_2}\gamma_{j_1, j_2} + \theta^1_{j_1}\Phi^2_{j_2}\alpha_{j_1, j_2} + \varphi^1_{j_1}\Theta^2_{j_2}\beta_{j_1, j_2} \\
& + \varphi^1_{j_1}\Phi^2_{j_2+1}\gamma_{j_1, j_2+1} + \theta^1_{j_1}\Phi^2_{j_2+1}\alpha_{j_1, j_2+1} \\
& + \varphi^1_{j_1}\Theta^2_{j_2+1}\beta_{j_1, j_2+1} + \varphi^1_{j_1+1}\Phi^2_{j_2}\gamma_{j_1+1, j_2} \\
& + \theta^1_{j_1+1}\Phi^2_{j_2}\alpha_{j_1+1, j_2} + \varphi^1_{j_1+1}\Theta^2_{j_2}\beta_{j_1+1, j_2} \\
& + \varphi^1_{j_1+1}\Phi^2_{j_2+1}\gamma_{j_1+1, j_2+1} + \theta^1_{j_1+1}\Phi^2_{j_2+1}\alpha_{j_1+1, j_2+1} \\
& + \varphi^1_{j_1+1}\Theta^2_{j_2+1}\beta_{j_1+1, j_2+1}, \\
p_4 = {} & \varphi^1_{j_1}\Theta^2_{j_2}\alpha_{j_1, j_2} + \varphi^1_{j_1}\Theta^2_{j_2+1}\alpha_{j_1, j_2+1} + \varphi^1_{j_1+1}\Theta^2_{j_2}\alpha_{j_1+1, j_2} \\
& + \varphi^1_{j_1+1}\Theta^2_{j_2+1}\alpha_{j_1+1, j_2+1}, \\
p_5 = {} & \theta^1_{j_1}\Phi^2_{j_2}\beta_{j_1, j_2} + \theta^1_{j_1}\Phi^2_{j_2+1}\beta_{j_1, j_2+1} + \theta^1_{j_1+1}\Phi^2_{j_2}\beta_{j_1+1, j_2} \\
& + \theta^1_{j_1+1}\Phi^2_{j_2+1}\beta_{j_1+1, j_2+1}, \\
p_6 = {} & \varphi^1_{j_1}\Phi^2_{j_2}\alpha_{j_1, j_2} + \varphi^1_{j_1}\Phi^2_{j_2+1}\alpha_{j_1, j_2+1} + \varphi^1_{j_1+1}\Phi^2_{j_2}\alpha_{j_1+1, j_2} \\
& + \varphi^1_{j_1+1}\Phi^2_{j_2+1}\alpha_{j_1+1, j_2+1}, \\
p_7 = {} & \varphi^1_{j_1}\Phi^2_{j_2}\beta_{j_1, j_2} + \varphi^1_{j_1}\Phi^2_{j_2+1}\beta_{j_1, j_2+1} + \varphi^1_{j_1+1}\Phi^2_{j_2}\beta_{j_1+1, j_2} \\
& + \varphi^1_{j_1+1}\Phi^2_{j_2+1}\beta_{j_1+1, j_2+1}.
\end{aligned}$$

The expressions for the coefficients in the remaining six regions can be derived similarly. ■

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Modeling and Recognition of Hand Gesture Using Colored Petri Nets

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***Abstract*— The main characteristics of human hand gestures can be summarized by their dynamic, multiattribute property. To utilize hand gestures as a way of interaction, it is necessary to analyze the motion patterns for each of the gesture attributes and finally to extract the whole interpretation by integrating the relevant factors across time.**

Previous research have shown the possibility for recognition of local aspects of hand gesture. But the global framework for finding the whole interpretation from the local aspects has yet to be provided.

In this article, we propose a colored Petri net model for high-level description of hand gestures. This model intercommunicates with simultaneous low-level recognizers and thus finds a whole-interpretation for the gesture.

***Index Terms*—Colored Petri net, hand gesture, hidden Markov model, modeling, recognition.**

I. INTRODUCTION

The use of hand gestures is often observed in everyday human conversation. Intrinsic three-dimensional (3-D) applications like virtual reality also began to incorporate more natural interaction methods such as speech and hand gestures in their interface.

In order to make hand gestures available in human-computer interaction, we should first be able to interpret the visual patterns

Manuscript received March 20, 1998; revised June 4, 1999. This work was supported in part by the Center for Artificial Intelligence Research and Samsung Advanced Institute of Technology.

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Publisher Item Identifier S 1083-4427(99)06988-X.