Modeling and Control of Fuzzy Discrete Event Systems

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***Abstract*—In order to make it possible to effectively represent deterministic uncertainties and vagueness as well as human subjective observation and judgement inherent to many real-world problems especially those in biomedicine, we introduce, in this paper, fuzzy states and fuzzy events and generalize (crisp) discrete event systems (DES) to fuzzy DES. The largely graph-based current framework of the crisp DES is unsuitable for the expansion and we have thus reformulated it using state vectors and event transition matrices which can be extended to fuzzy vectors and matrices by allowing their elements to take values between 0 and 1. To measure information related to fuzzy DES, we generalize the crisp DES observability. The new observability allows one to determine whether or not the system output observed is sufficient for decision making. Finally, we extend optimal control of DES to fuzzy DES. The new fuzzy DES theory is consistent with the existing theory, both at conceptual and computation levels, in that the former contains the latter as a special case when the memberships must be either 0 or 1. Numerical examples are provided to illustrate the theoretical development.**

***Index Terms*—Controllability, discrete event systems, fuzzy systems, observability, optimal control.**

I. Introduction

THERE exist countless (complex) systems that cannot be effectively described, at a higher level, by differential equations, but can be by traces (or sequences) of events that record significant qualitative changes in the state of the system. These states are logical or symbolic rather than numerical. These systems can be described as discrete event systems (DES) whose behavior consists of sequence of occurrences of distinct events. These events, for instance, can describe sending or receiving messages in computer networks, or processing a part in a manufacturing plant. Comprehensive study and development of DES theory is a recent endeavor. Only after the proliferation of complex systems such as computer systems and networks, have DES been systematically studied. Even though DES are quite different from traditional continuous variable dynamic systems, they clearly involve objectives of control and optimization. DES theory addresses the issues of modeling, control, and optimization of DES [1], [9], [11]. It has been applied to practical systems such as automated manufacturing systems, database concurrency control, feature interactions in telecommunication networks, protocol verification and synthesis in communication networks, and protocol conversion and gateway synthesis in computer networks.

To date, only DES with crisp states and events have been extensively studied in the literature. Although seemingly sufficient for many application domains, crisp DES are not adequate for some important fields. This is especially true when we consider biomedical applications in which the state and state transition of a system (e.g., a person's health status) are always somewhat uncertain and vague even in a deterministic sense. Subjective human observation, judgement and interpretation (e.g., by a physician or a patient) invariably play a significant role in describing the status of state, usually not crisp. For instance, it is vague when a patient's health is said to be "good." Furthermore, the transition from one state to another is also vague. It is hard to say how exactly a patient's condition has changed from "good" to "bad."

Fuzzy set is an extension of the classical set [15], [16]. In traditional set theory, an object either completely belongs to a set or does not at all. No partial membership is allowed. Nevertheless, in our daily lives, there exist countless vague and subjective concepts that we humans can easily describe, understand and communicate with each other but conventional mathematics fails to handle in a rational way. The concept "young" is an example to the point. A person's age is numerically precise. However, relating a particular age to "young" can be difficult and confusing. What age is young and what age is not? The nature of this question to any particular person is deterministic, not stochastic. To address the issues like this, fuzzy set theory generalizes 0 and 1 membership values of a classical set to a membership function of a fuzzy set ranging from 0 to 1; 0 means no association, 1 indicates complete association, and any number in between means partial association. One possible fuzzy set "young" is provided in Fig. 1, along with a classical counterpart for comparison. According to the fuzzy set, every age is "young" to a degree. Of course, no standard fuzzy set "young" exists. Not only do different people have different fuzzy sets for the same concept, even for the same person, different fuzzy sets are used when the context in which the concept involves varies [14].

Fuzzy sets and fuzzy logic can quantitatively and effectively represent and compute deterministic uncertainty as well as subjective and qualitative concepts. Such representation and calculation are mathematically precise. Fuzzy set theory is complement to probability and statistics theory because the latter handles largely randomness and stochastic uncertainty whereas the former can deal with possibility and subjectivity. Deterministic vagueness, imprecision, uncertainty, and subjectivity are the norm for most of the complex information and systems in

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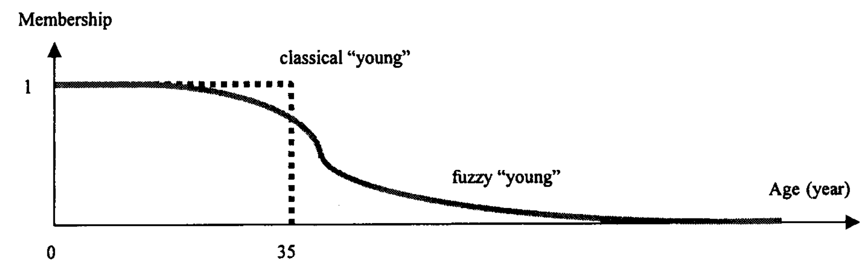

Fig. 1. Examples of fuzzy set "young" and classical set "young."

existence, including medical information (e.g., the way that a doctor describes the diagnosis, treatment and prognosis of a disease). Thus, fuzzy sets and fuzzy logic are necessary.

Although definitions of fuzzy state, fuzzy event and fuzzy au tomata were developed in as early as the late 1960s [13], research has not been active since [6]. A comprehensive theory of fuzzy DES are yet to be established, so are many fundamentally important concepts, methods, and theorems in the traditional DES, such as controllability, observability and optimal control.

The present paper attempts to establish a framework for a comprehensive theory of fuzzy DES by combining fuzzy set theory with DES. To model fuzzy DES, we generalize the conventional (crisp) finite automaton model to fuzzy finite automaton model. This model is most convenient and simple. It also allows one to first build each component model separately. The overall system model can then be synthesized automatically (by a computer program) from them by parallel composition

Controllability and observability are studied in supervisory control of crisp DES [9], [11]. They are used to describe conditions for the existence of a supervisor. We will generalize crisp observability, valued either 0 or 1, to fuzzy observability of $[0, 1]$, capable of describing the extent to which the output of a system contains sufficient information to make a right decision. We will discuss this generalized observability in the context of both crisp DES and fuzzy DES.

To control a DES, two mechanisms exist: disablement and enforcement. Our optimal control objective for fuzzy DES is to maximize their effectiveness and minimize cost. We will use an online scheme to design a controller that achieves our optimal control objective. This is an extension of the approach developed for crisp DES.

The rest of the paper is organized as follows. In Section II, we will review the background for crisp DES and provide a motivating example. In Section III, we will extend crisp finite automata to fuzzy automata to model fuzzy DES. In Section IV, we will investigate observability of both crisp DES and fuzzy DES. In Section V, we will study optimal control of fuzzy DES. Some of our relevant preliminary results appear in [10].

II. Background in Discrete Event Systems

In crisp DES theory, finite automata are used to model discrete event systems. A crisp finite automaton consists of a set of discrete states Q, a set of events Σ, and a transition mapping $\delta : Q \times \Sigma \to Q$ that describes what event can occur at which state and the resulting new state. Assuming the initial state is q_0, a finite automaton is denoted by

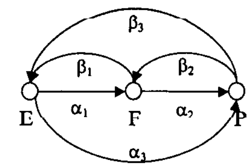

Fig. 2. Finite automaton to model lung condition.

$$G = (Q, \Sigma, \delta, q_o).$$

Let us consider the use of finite automaton in medical applications. For example, if one classifies a patient's lung condition as excellent (E), fair (F) and poor (P), then the corresponding crisp finite automaton is shown as in Fig. 2. In the figure, α_i and β_i denote the events describing the patient's health condition is deteriorating and improving, respectively. We assume for now that both the states and events are crisp. In this finite automaton

$$Q = \{E, F, P\};$$
$$\Sigma = \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\};$$
$$q_o = E;$$
δ is shown in the figure.

In application, each component of a DES is first modeled by a finite automaton G_i. The overall system is then synthesized by a parallel composition $\|$. The parallel composition of two finite automata G_1 and G_2 is defined as follows. For

$$G_1 = (Q_1, \Sigma_1, \delta_1, q_{1o})$$
$$G_2 = (Q_2, \Sigma_2, \delta_2, q_{2o})$$

the parallel composition is

$$G_1 \| G_2 = (Q_1 \times Q_2, \Sigma_1 \cup \Sigma_2, \delta_1 \times \delta_2, (q_{1o}, q_{2o}))$$

where $\delta_1 \times \delta_2 : (Q_1 \times Q_2) \times (\Sigma_1 \cup \Sigma_2) \to (Q_1 \times Q_2)$ is defined as

$$(\delta_1 \times \delta_2)((q_1, q_2), \sigma) = \begin{cases} (\delta_1(q_1, \sigma), q_2), & \text{if } \sigma \in \Sigma_1 \backslash \Sigma_2 \\ (q_1, \delta_2(q_2, \sigma)), & \text{if } \sigma \in \Sigma_2 \backslash \Sigma_1 \\ (\delta_1(q_1, \sigma), \delta_2(q_2, \sigma)), & \text{if } \sigma \in \Sigma_1 \cap \Sigma_2 \end{cases}$$

The notation $\Sigma_1 \backslash \Sigma_2$ has the same meaning as Σ_1 / Σ_2 used in some of the literature. For illustration purpose, let us use another

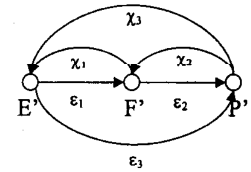

Fig. 3. Finite automaton to model heart condition.

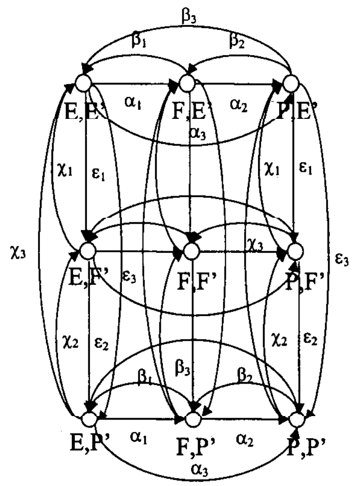

Fig. 4. Combining heart and lung conditions by parallel composition.

finite automaton, shown in Fig. 3, to model the heart condition of the same patient.

In the figure, E', F' and P' mean excellent, fair, and poor, respectively. ε_i and χ_i denote the events that the patient's heart condition is deteriorating and improving, respectively. The parallel composition $G_1 \parallel G_2$ represents the combined heart and lung condition of the patient, which is shown in Fig. 4.

Obviously, this parallel composition can be easily extended to more than two finite automata. Usually, the parallel composition is implemented by a computer program so that the synthesis of the overall system can be done automatically. This automation makes the synthesis of complex systems feasible.

III. Fuzzy Discrete Event Systems

A. Reformulation of the Representation of Crisp Finite Automata

As illustrated in the previous section, DES modeled by finite automata are currently represented graphically by nodes and arcs, which is unsuitable for the extension to fuzzy DES. To introduce fuzzy states and fuzzy events in DES models, we need to reformulate crisp finite automata first. We propose the following alternative representation of automata so that it can easily incorporate fuzzy logic and fuzzy sets. To this end, let us enumerate the states of G as $1, 2, \ldots, n$ ($n = |Q|$, the number of states in G). The current state of the system is represented by a vector

$$q = [m_k]_{1\times n} = [m_1 \quad \cdots \quad m_n]$$

where $m_k = 1$ (and $m_i = 0$ for $i \neq k$) to indicate that the system is currently in state k. The transition function δ is represented by h matrices ($h = |\Sigma|$, the number of events in G), one for each event. For example, the transition labeled by α is represented by matrix

$$\alpha = [\alpha_{ij}]_{n\times n} = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \cdots & & \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix}$$

where α_{ij} equals 1 if α is defined from state i to state j; and equals 0 otherwise. In this way, if the current state of the system is q and α occurs in the system, then the next state will be $q\alpha$ in terms of matrix multiplication. Note that we use q to denote both a state and the vector representing the state. Similarly, we use α to denote both an event and the matrix representing the event transition.

For the automaton in Fig. 2, let us enumerate state as 1 (for state E), 2 (for state F) and 3 (for state P). The initial state is then represented by

$$q_0 = [1 \quad 0 \quad 0].$$

The events and their transitions are represented by

$$\alpha_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \beta_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\alpha_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \beta_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\alpha_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \beta_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

So, if β_1 occurs at state F (represented by $[0 \ 1 \ 0]$), then the next state is

$$[0 \quad 1 \quad 0] \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [1 \quad 0 \quad 0],$$

which represents state E.

B. Fuzzy States, Fuzzy Events, and Fuzzy Finite Automata

Now that numerical vectors and matrices are used to represent crisp states and events, we utilize this format to develop fuzzy DES as follows.

We first allow the elements of state vector q to have values between 0 and 1, as opposed to 0 or 1. In this way, $\tilde{q} = [\tilde{m}_k]_{1\times n}$ is a fuzzy state instead of crisp state, permitting partial membership. For example, a patient's condition can simultaneously belong to "Excellent" with a membership 0.4 and "Fair" with a membership 0.8, which is represented by the vector

$$\tilde{q} = [0.4 \quad 0.8 \quad 0].$$

Similarly, we allow the elements of event transition matrix $\tilde{\alpha} = [\tilde{\alpha}_{ij}]_{n\times n}$ to take values between 0 and 1 so that fuzzy events (i.e., events with partial memberships) can be represented. For example, $\tilde{\alpha}_{ij} = 0.6$ indicates that the possibility for the system to transit from state i to state j when event $\tilde{\alpha}$ occurs is 0.6. Unlike crisp (deterministic DES), an event in fuzzy DES may take the system to more than one states with different degrees. For example, a treatment may change a patient's condition into both excellent and fair with different degrees.

With these extensions, if the current state vector is q and event α occurs, then the next state vector is $\tilde{q}' = \tilde{q} \circ \tilde{\alpha}$, where $\circ$ is the following max-product operation in fuzzy set theory [7]:

$$[P \circ R](x, z) = \max \text{prod}[P(x, y), R(y, z)].$$

Let us see how to generalize the example automaton in Fig. 2 to fuzzy finite automaton. Let us assume that α_1 is a fuzzy event defined by

$$\tilde{\alpha}_1 = \begin{bmatrix} 0.1 & 0.9 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0 & 0.1 & 0.1 \end{bmatrix}.$$

Now, if the current state is

$$\tilde{q} = [0.4 \quad 0.8 \quad 0]$$

then the next state after α_1 is calculated as the first equation shown at the bottom of the page. In summary, a fuzzy DES can be represented by a fuzzy automaton

$$\tilde{G} = (\tilde{Q}, \tilde{\Sigma}, \tilde{\delta}, \tilde{q}_0)$$

where $\tilde{Q}, \tilde{\Sigma}, \tilde{\delta}, \tilde{q}_0$ are fuzzy extensions of the crisp Q, Σ, δ, q_o respectively. Formally, $\tilde{Q}$ is a set of fuzzy states $\tilde{q}$, where $\tilde{q}$ is represented by a vector over $[0, 1]$; $\tilde{\Sigma}$ is a set of fuzzy events $\tilde{\alpha}$, where $\tilde{\alpha}$ is represented by a matrix over $[0, 1]$; $\tilde{\delta}{:}\tilde{Q} \times \tilde{\Sigma} \rightarrow \tilde{Q}$ is a transition function, where $\tilde{\delta}(\tilde{q}, \tilde{\alpha}) = \tilde{q} \circ \tilde{\alpha}$; and $\tilde{q}_0$ is the initial state.

The parallel composition can also be extended to fuzzy automata as follows. For

$$\tilde{G}_1 = (\tilde{Q}_1, \tilde{\Sigma}_1, \tilde{\delta}_1, \tilde{q}_{1o})$$
$$\tilde{G}_2 = (\tilde{Q}_2, \tilde{\Sigma}_2, \tilde{\delta}_2, \tilde{q}_{2o})$$

the parallel composition is denoted by

$$\tilde{G}_1 \parallel \tilde{G}_2 = (\tilde{Q}_1 \times \tilde{Q}_2, \tilde{\Sigma}_1 \cup \tilde{\Sigma}_2, \tilde{\delta}_1 \times \tilde{\delta}_2, (\tilde{q}_{1o}, \tilde{q}_{2o}))$$

Here the membership of state $(\tilde{q}_i, \tilde{q}_j)$ is given by

$$\tilde{m}_{ij} = \tilde{m}_i^1 \tilde{\cap} \tilde{m}_j^2 = \min\{\tilde{m}_i^1, \tilde{m}_j^2\}$$

where $\tilde{m}_i^1$ is the membership of $\tilde{q}_i$ in $\tilde{G}_1$ and $\tilde{m}_j^2$ is the membership of $\tilde{q}_j$ in $\tilde{G}_2$ whereas $\tilde{\cap}$ denotes a fuzzy AND operator. There are dozens of such operator and we choose to use one of the most popular and effective ones—Zadeh fuzzy AND operator. Similarly, for an event α, the possibility of $\tilde{G}_1 \parallel \tilde{G}_2$ transiting from $(\tilde{q}_i, \tilde{q}_j)$ to $(\tilde{q}_u, \tilde{q}_v)$ is given by the second equation shown at the bottom of the page, where $\tilde{\alpha}_{iu}^1$ is the possibility of $\tilde{G}_1$ transiting from $\tilde{q}_i$ to $\tilde{q}_v$ as defined early and $\tilde{\alpha}_{jv}^2$ is similar.

For example, let us consider the parallel composition of two fuzzy automata as illustrated in Figs. 2 and 3, denoted by $\tilde{G}_1$ and $\tilde{G}_2$ respectively. The dimension of the state vector is $3 \times 3 = 9$. That is

$$\begin{aligned}\tilde{q} &= [\tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_4, \tilde{q}_5, \tilde{q}_6, \tilde{q}_7, \tilde{q}_8, \tilde{q}_9] \\ &= [EE', EF', EP', FE', FF', FP', PE', PF', PP'].\end{aligned}$$

If in $\tilde{G}_1$ the event $\tilde{\alpha}_1$ is described by matrix

$$\tilde{\alpha}_1 = \begin{bmatrix} 0.1 & 0.9 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0 & 0.1 & 0.1 \end{bmatrix}$$

then in $\tilde{G}_1 \parallel \tilde{G}_2$, it is represented by matrix

$$\tilde{\alpha}_1 = \begin{bmatrix} 0.1 & 0 & 0 & 0.9 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0.9 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0.1 & 0 & 0 & 0.9 & 0 & 0 & 0.1 \\ 0.1 & 0 & 0 & 0.1 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0.1 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0.1 & 0 & 0 & 0.1 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0.1 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0.1 \end{bmatrix}.$$

IV. Observability of Fuzzy Discrete Event Systems

A. New Formulation of Observability for Crisp Discrete Event Systems

Not all events in a DES modeled by G are observable by a controller or decision maker (due to the lack of sensors or detection means, too high cost of detection, or difficulty in information transmission). In crisp DES, the set of observable events is denoted by Σ_o and the set of unobservable events by Σ_{uo}. The

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$$\begin{aligned}\tilde{q}' = \tilde{q} \circ \tilde{\alpha}_1 &= [0.4 \quad 0.8 \quad 0] \circ \begin{bmatrix} 0.1 & 0.9 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0 & 0.1 & 0.1 \end{bmatrix} \\ &= [\max\{0.04, 0.08, 0\} \quad \max\{0.36, 0.08, 0\} \quad \max\{0.04, 0.08, 0\}] = [0.08 \quad 0.36 \quad 0.08].\end{aligned}$$

$$\tilde{\alpha}_{(i,j)(u,v)} = \begin{cases} \min\left\{\tilde{\alpha}_{iu}^1, \tilde{\alpha}_{jv}^2\right\}, & \text{if } \tilde{\alpha} \text{ is an event in both } \tilde{G}_1 \text{ and } \tilde{G}_2 \\ \tilde{\alpha}_{iu}^1, & \text{if } \tilde{\alpha} \text{ is not an event in } \tilde{G}_2 \text{ and } j = v \\ \tilde{\alpha}_{jv}^2, & \text{if } \tilde{\alpha} \text{ is not an event in } \tilde{G}_1 \text{ and } i = u \\ 0, & \text{otherwise} \end{cases}$$

key question to observability is following: By observing the occurrences of observable events, does a decision maker have sufficient information to make a correct decision? The answer to this question depends naturally on what kind of decision to be made.

We must reformulate the representation of the crisp observability before introducing observability of fuzzy DES. To make our approach general, we will not limit ourselves to any particular decision making. Instead, we will only require that the decision making be state dependent. Mathematically, this can be represented by a state feedback $\phi: Q \rightarrow \Phi$, for some (problem dependent) Φ [12]. This general definition of ϕ can represent many control action of interest (e.g., disablement, enablement, or enforcement). For a decision maker to make a correct decision, it must have sufficient information available to distinguish two states if the decisions at these two states are different. Such information is provided by the observation of occurrences of observable events in the system. Since not all events are observable to the decision maker, from a state q, some unobservable events may occur to take the system to another state q'' which is undetected by the decision maker. We call the set of all such q'' the unobservable reach of q [1], [5]. The unobservable reach can be calculated from q and $\Sigma_{\rm uo}$ as follows. Suppose that the current state is represented by vector $q = [0 \ \cdots \ 1 \ \cdots \ 0]$ and α is not observable, then the unobservable reach consists of state $q, q\alpha$ (if α occurs once), $q\alpha\alpha$ (if α occurs twice), $\ldots$, up to $q\alpha^n$ (n is the number of states in G; after n the system will loop back to some previously visited states). If there are more than one unobservable events, then the unobservable reach can be calculated as $q\Sigma_{\rm uo}^{\leq n}$, where $\Sigma_{\rm uo}^{\leq n}$ denotes all possible sequences of $\Sigma_{\rm uo}$ with length less than n. Because of the unobservable reach, the decision maker may not know exactly which state the system is in. To represent the estimate of the current state by the decision maker, we use a state estimate vector r. The (current) state estimate is defined to be the set of all possible states the system may be in. The difference between q and r is that we allow r to have more than one nonzero elements. For example, $r = [1 \ 1 \ 0 \ \cdots \ 0]$ denotes that the system could be in either state 1 or state 2. Using this representation, we can calculate the state estimate as follows.

Proposition 1: Consider a crisp DES modeled by $G = (Q, \Sigma, \delta, q_o)$. Assume that the observable and unobservable events are Σ_o and $\Sigma_{\rm uo}$ respectively. The state estimate can be calculated recursively as follows.

1) Initially, the estimate is $r_o = q_o\Sigma_{\rm uo}^{\leq n}$.
2) If the current estimate is r, then after the occurrence of an observable event α, the next estimate is $r' = r\alpha\Sigma_{\rm uo}^{\leq n}$.

Proof:

1) It follows the definition of unobservable reach and the fact that the system is initially at q_o.
2) If the current estimate is r, then after the occurrence of α the possible states are $r\alpha$. Unobservable reach brings the estimate to $r\alpha\Sigma_{\rm uo}^{\leq n}$. ■

We denote the set of all possible state estimates of G by $R(G)$. To define observability, we need to define the consistency of decisions at different states. We do this by using a consistency matrix $W = [w_{ij}]_{n \times n}$, where w_{ij} is 0 if the decision in states i and j are consistent (meaning no conflict), and 1 otherwise. For example, if a doctor's decision to treat patients with excellent and fair conditions are consistent but different from the treatments of patients with poor conditions, then

$$W = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Obviously, W depends on a particular decision under consideration. Under this definition, the ambiguity in state estimate r is inconsequential to the decision maker if and only if $rWr^T = 0$. Therefore, we have the following definition.

Definition 1: A crisp DES modeled by $G = (Q, \Sigma, \delta, q_o)$ is said to be observable with respect to consistency matrix W if for all possible state estimates $r \in R(G), rWr^T = 0$.

This observability depends not only on the DES and the set of observable events, but also on the particular decision to make, which is represented by the consistency matrix W.

Next, we would like to introduce a stronger version of observability that is independent of W. To this end, we consider the case where decisions are different in every state. Therefore, let W^o be the consistency matrix where all the elements are 1 except the diagonal elements that are all 0. We have the following.

Definition 2: A crisp DES modeled by $G = (Q, \Sigma, \delta, q_o)$ is said to be strongly observable if it is observable with respect to W^o.

Clearly, strong observability implies observability for any W, leading to the following.

Proposition 2: If a DES is strongly observable, then it is observable with respect to any consistency matrix W.

Proof: For any consistency matrix W, its diagonal elements must be 0 because the decision in any state cannot be inconsistent with itself. By the definition of W^o, $W \leq W^o$, for all W. Therefore, it is not difficult to see that for any r, $rWr^T \leq rW^or^T$. In other words, $rW^or^T = 0$ implies $rWr^T = 0$. ■

For a crisp DES, it is unlikely that the system will be strongly observable. However, the strong observability makes a lot of sense if we consider fuzzy DES.

B. Definition and Computation of Observability for Fuzzy Discrete Event Systems

To extend observability to fuzzy DES, we allow both state q and event α to be fuzzy as described in Section 3. We also allow an event to be partially observable (or partially unobservable). In other words, we associate each event with degrees of observability and unobservability

$$d_\alpha = (d_\alpha^o, d_\alpha^{\rm uo})$$

where d_α^o is the degree of observability and $d_\alpha^{\rm uo}$ the degree of unobservability of event α. We require that

$$d_\alpha^o + d_\alpha^{\rm uo} = 1.$$

With this extension, we no longer distinguish observable events and unobservable events: every event is observable (or unobservable) to certain degree.

To calculate the state estimate for fuzzy DES, we generalize the method described above for crisp DES. Obviously, now the state estimate $\tilde{r}$ can take values between 0 and 1, indicating the confidence of the decision maker on the system to be in any particular state. If the current estimate is r and α is partially observed, then the next state estimate is

$$\tilde{r}' = \tilde{r} \circ (d_\alpha^o \cdot \tilde{\alpha} \tilde{\cup} d_\alpha^{uo} \cdot \tilde{\alpha}^u)$$

where $\tilde{\cup}$ represents a fuzzy OR operator and we use Zadeh OR operator (taking maximal). Here, α^u is only a symbolic notation to denote a fuzzy event, and the superscription is not an exponential index. This fuzzy event is computed as: $\tilde{\alpha} = I\tilde{\cup}\tilde{\alpha}\tilde{\cup}\tilde{\alpha}^2 \cdots \tilde{\cup}\tilde{\alpha}^n$, where n is the number of states. The exponential calculation of $\tilde{\alpha}^n$ is $\tilde{\alpha}^n = \tilde{\alpha} \otimes \tilde{\alpha} \cdots \tilde{\alpha} \otimes \tilde{\alpha}$, where $\otimes$ is the standard composition of fuzzy relations (e.g., [7])

$$[P \otimes R](x, z) = \max \min[P(x, y), R(y, z)].$$

To illustrate these theoretical developments concretely, let us continue the example for automaton in Fig. 2. Suppose that the event β_1 has

$$d_{\tilde{\alpha}_1} = \left(d_{\tilde{\alpha}_1}^o, d_{\tilde{\alpha}_1}^{uo}\right) = (0.2, 0.8)$$

and the current state estimate is

$$\tilde{r} = [0.3 \quad 0.7 \quad 0.1].$$

Then the next state estimate is computed

$$\tilde{r}' = \tilde{r} \circ \left(d_{\tilde{\alpha}_1}^o \cdot \tilde{\alpha}_1 \tilde{\cup} d_{\tilde{\alpha}_1}^{uo} \cdot \tilde{\alpha}_1^u\right)$$

where

$$\begin{aligned}
\tilde{\alpha}_1^u &= I\tilde{\cup}\tilde{\alpha}_1\tilde{\cup}\tilde{\alpha}_1^2\tilde{\cup}\tilde{\alpha}_1^3 \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tilde{\cup} \begin{bmatrix} 0.1 & 0.9 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0 & 0.1 & 0.1 \end{bmatrix} \\
&\quad \tilde{\cup} \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \end{bmatrix} \tilde{\cup} \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0.9 & 0.1 \\ 0.1 & 1 & 0.1 \\ 0.1 & 0.1 & 1 \end{bmatrix}.
\end{aligned}$$

Hence

$$\begin{aligned}
\tilde{r}' &= \tilde{r} \circ \left(d_{\tilde{\alpha}_1}^o \cdot \tilde{\alpha}_1 \tilde{\cup} d_{\tilde{\alpha}_1}^{uo} \cdot \tilde{\alpha}_1^u\right) \\
&= [0.3 \quad 0.7 \quad 0.1] \circ \begin{bmatrix} 0.8 & 0.72 & 0.08 \\ 0.08 & 0.8 & 0.08 \\ 0.08 & 0.08 & 0.8 \end{bmatrix} \\
&= [0.24 \quad 0.56 \quad 0.08].
\end{aligned}$$

To extend the observability of crisp DES to fuzzy observability of fuzzy DES, we first introduce partial consistency by allowing the consistency matrix $W = [w_{ij}]_{n \times n}$ to take values between 0 and 1. Therefore, if $w_{ij} = 0.7$, then the degree of inconsistency between states i and j is 0.7. For a state estimate $\tilde{r}$, $\tilde{r} \circ W \circ \tilde{r}^T$, describes the degree of consistency among states in r (0 being most consistent and 1 being least).

In the example of Fig. 2, if

$$W = \begin{bmatrix} 0 & 0.7 & 0.3 \\ 0.7 & 0 & 0.6 \\ 0.3 & 0.6 & 0 \end{bmatrix}$$

and

$$\tilde{r} = [0.3 \quad 0.7 \quad 0.1]$$

then

$$\begin{aligned}
\tilde{r} \circ W \circ \tilde{r}^T &= [0.3 \quad 0.7 \quad 0.1] \\
&\circ \begin{bmatrix} 0 & 0.7 & 0.3 \\ 0.7 & 0 & 0.6 \\ 0.3 & 0.6 & 0 \end{bmatrix} \circ \begin{bmatrix} 0.3 \\ 0.7 \\ 0.1 \end{bmatrix} = 0.334.
\end{aligned}$$

Let $R(G)$ be the set of all possible state estimates of the fuzzy DES G. We can now use the following measure, which we call the observability measure (OM), to measure the degree of observability for fuzzy DES

$$\text{OM}(W) = 1 - \frac{\sum_{r \in R(G)} \tilde{r} \circ W \circ \tilde{r}^T}{\sum_{r \in R(G)} \tilde{r} \circ \tilde{r}^T}.$$

If $\text{OM}(W) = 1$, then the fuzzy DES is completely observable, and consistent decisions can be made based on the observation. That is, the availability of the information is sufficient for the particular decision specified by W. Similar to crisp DES, we can also introduce a strong measure of observability (SOM) by taking $W = W^\circ$. Since this SOM is independent of any particular decision making, it represents the inherent information structure of the fuzzy DES.

V. Optimal Control of Fuzzy Discrete Event Systems

In this section, we study the control of fuzzy DES. Our goal is to design a control that is optimal and effective. In traditional DES control, two control mechanisms exist [8].

1) Disablement: Events in $\Sigma_c \subseteq \Sigma$ can be disabled by the controller. They are called controllable events.
2) Enforcement: Events in $\Sigma_f \subseteq \Sigma$ can be enforced by the controller. They are called enforceable events.

Formally, the controller is given by two state feedbacks

$$C_d: \tilde{Q} \to 2^{\Sigma_c}$$
$$C_f: \tilde{Q} \to 2^{\Sigma_f}$$

where $C_d(\tilde{q})$ is the set of events to be disabled at $\tilde{q}$ (that is, they are not allowed to occur) and $C_f(\tilde{q})$ is the set of events to be enforced at $\tilde{q}$ (that is, one of them is forced to occur).

Note that the control mechanisms are crisp in that if an event can be disabled, then its occurrence can be prevented with certainty. Similarly, if an event is enforced, then it will definitely occur. We assume in the present section that all events are observable (otherwise, we will need to study observability as in the previous section).

Our control objective is to minimize cost and/or maximize effectiveness. Therefore, for each fuzzy state vector q, we derive an effectiveness measure $M(\tilde{q})$ and a cost $C(\tilde{q})$. For example,

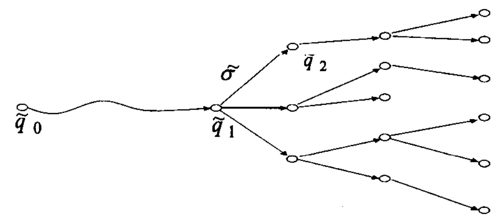

Fig. 5. Forward-looking tree for online control synthesis.

in the patient's health condition model of Fig. 4, let the state vector be

$$\tilde{q} = [\tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_4, \tilde{q}_5, \tilde{q}_6, \tilde{q}_7, \tilde{q}_8, \tilde{q}_9]$$
$$= [EE', EF', EP', FE', FF', FP', PE', PF', PP'].$$

If we intend to use effectiveness to describe how good a patient's overall condition is, then we can use a weighted sum that gives more weight on E (excellent) such as

$$E(\tilde{q}) = EE' + 0.6EF' + 0.5FE' + 0.3FF'.$$

For example, if

$$\tilde{q} = [0.1 \quad 0.2 \quad 0 \quad 0.1 \quad 0.3 \quad 0 \quad 0 \quad 0.1 \quad 0.4]$$

then

$$E(\tilde{q}) = 0.1 + 0.6 \times 0.2 + 0.5 \times 0.1 + 0.3 \times 0.3 = 0.36.$$

Similarly, we can use cost to describe how poor a patient's overall condition is

$$C(\tilde{q}) = PP' + 0.7PF' + 0.6FP' + 0.3EP' + 0.2PE'.$$

For example, for above $\tilde{q}$

$$E(\tilde{q}) = 0.4 + 0.7 \times 0.1 + 0.6 \times 0 + 0.3 \times 0 + 0.2 \times 0$$
$$= 0.47.$$

In general, the effectiveness M and cost C are specified as functions

$$M : V \rightarrow Z$$
$$C : V \rightarrow Z$$

where Z is a set of nonnegative real numbers and $V = [0,1]^n$ is the set of all possible values that a state vector can take. The actual definition of $M(\cdot)$ and $C(\cdot)$ depends on the particular problem at hand.

In crisp DES, the design of a control can be done online. That is, after each occurrence of events, the controller will evaluate the possible future execution of the system and determine which events to disable and which events to enforce. Therefore, we will construct a forward-looking tree as shown in Fig. 5.

For fuzzy DES, the way to construct this tree is different from the construction in crisp DES [3]. Each node in the tree no longer represents a crisp state, but rather a fuzzy state with vector $\tilde{q} = [\tilde{m}_1 \ \cdots \ \tilde{m}_n]$. The root of the tree represents the current state vector, denoted by $\tilde{q}_1$. We then check if an event σ is possible at $\tilde{q}_1$. That is, we calculate $\tilde{q}_1 \circ \tilde{\sigma}$. If $\tilde{q}_1 \circ \tilde{\sigma} \neq 0$, then $\tilde{\sigma}$ is possible and will lead to a new node denote by $\tilde{q}_2 = \tilde{q}_1 \circ \tilde{\sigma}$. In this way, we can construct the forward-looking tree for fuzzy DES. How big the tree is depends on the number of forward-looking steps. To look ahead more means better results but more computation. Also, a variable lookahead policy can be used by specifying some intelligent terminating conditions [4]. Since these conditions are equally applicable to fuzzy DES, we will not discuss them further.

The main difference between a crisp DES tree and the corresponding fuzzy DES tree is that there are more branches in the fuzzy DES tree. The reason is that since the system can now be partially in more than one state, more events are possible at each node in the fuzzy DES tree.

After constructing the forward-looking tree, we can calculate the effectiveness and cost of each nodes according to $M(\cdot)$ and $C(\cdot)$. We define the effectiveness and cost of a branch τ in the tree to be the effectiveness and cost of its terminal node. That is, if branch τ leads to node $\tilde{q}$, then $M(\tau) = M(\tilde{q})$ and $C(\tau) = C(\tilde{q})$.

Based on this formalism, several optimization problems can be studied. The first optimization problem is to maximize the effectiveness for a given cost. In other words

$$\max_{T(\tilde{q})} M(\tau), \text{ such that } C(\tau) < L$$

where $T(\tilde{q})$ is the forward-looking tree from $\tilde{q}$ as shown in Fig. 5 and L is a (given) constant restricting the maximum cost. To interpret this problem correctly, we must say what we mean by maximizing the effectiveness. In this paper, we interpret it as maximizing the effectiveness some time in the future, that is, at some nodes in $T(\tilde{q})$. Clearly, other interpretations are also possible and we will leave them for future research. On the other hand, we interpret $C(\tau) < L$ as the cost being less than L all the time. This is appropriate if cost describes toxicity, side effects, and/or complications in medical treatments.

With the above interpretations, we can solve the first optimization problem as follows. In the forward-looking tree $T(\tilde{q})$, let us first identify all nodes whose cost is greater than L and denote these "undesired nodes" as

$$UN(\tilde{q}) = \{\tilde{q}' \in T(\tilde{q}) : C(\tilde{q}) \geq L\}.$$

Obviously, our control must ensure that the system will never enter these undesired nodes. This corresponds to the safety problem in conventional supervisory control theory. To maximize the effectiveness, we identify a node or nodes whose effectiveness is largest among all the nodes in $T(\tilde{q})$. We denote these nodes as "most effective nodes"

$$MN(\tilde{q}) = \{\tilde{q}' \in T(\tilde{q}) : (\forall \tilde{q}'' \in T(\tilde{q})) E(\tilde{q}'') \leq E(\tilde{q}')\}.$$

We would like the system to reach these marked nodes, which is a liveness problem in conventional supervisory control theory. Therefore, after defining $UN(\tilde{q})$ and $MN(\tilde{q})$, the first optimization problem can be solved by solving a standard supervisory control problem of safety and liveness with $UN(\tilde{q})$ as the illegal states and $MN(\tilde{q})$ as the marked states [2], [8]. Since the control synthesis for such a problem has been discussed in the literature, we will not discuss it further, except to note that the solution exists if the corresponding language is controllable [8].

If the language is not controllable, then the largest effectiveness value is not achievable and we must add nodes with the second largest effectiveness value to $MN(\tilde{q})$ and try to solve the safety and liveness problem again until a solution is found.

The second optimization problem is to minimize the cost for a given effectiveness, that is

$$\min_{T(\tilde{q})} C(\tau), \text{ such that } M(\tau) > K.$$

We interpret the minimization as minimizing the cost some time in the future and $M(\tau) > K$ as the effectiveness being greater than K all the time. This second optimization problem is therefore dual to the first optimization problem. We define the undesired nodes and minimal cost nodes as

$$UN'(\tilde{q}) = \{\tilde{q}' \in T(\tilde{q}) : M(\tilde{q}) \leq K\},$$

and

$$MN'(\tilde{q}) = \{\tilde{q}' \in T(\tilde{q}) : (\forall \tilde{q}'' \in T(\tilde{q}))C(\tilde{q}'') \geq C(\tilde{q}')\}.$$

The second optimization problem is again translated into the problem of designing a safe and live controller with illegal state $UN'(\tilde{q})$ and $MN'(\tilde{q})$.

VI. Conclusion

We have extended crisp DES to fuzzy DES by introducing partial memberships to both states and (event) transitions. We have studied observability of fuzzy DES and proposed an observability measure. We have also investigated some optimal control problems and found a way to translate the problems into the problem of ensuring safety and liveness, whose solution has been obtained before. We believe that fuzzy DES theory has many applications, especially in medical fields. We are currently applying fuzzy DES to disease treatment with medical doctors.

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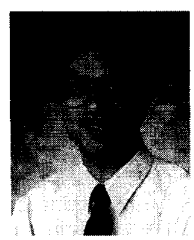

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