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Analytical structure of a two-input two-output fuzzy controller and its relation to PI and multilevel relay controllers

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Abstract: This study investigates the analytical structure of a two-input two-output fuzzy controller which employs triangular-shaped input fuzzy sets, trapezoidal-shaped output fuzzy sets, linear control rules, probabilistic AND fuzzy logic, Łukasiewicz OR fuzzy logic, Mamdani's minimum inference method and the center of gravity defuzzification algorithm. We analytically prove that the structure of the fuzzy controller is the sum of global four-dimensional multilevel relays and local nonlinear proportional-integral (PI) controllers with variable gains continuously changing with process outputs. The global multilevel relays play a major role in determining control action of the fuzzy controller while the local PI controllers locally fine-tune the control action of the relays. Properties of the fuzzy controller structure are analytically and quantitatively investigated. Moreover, it is proved that, as the number of control rules approaches ∞, each global four-dimensional multilevel relay becomes the sum of two global linear PI controllers while the local PI controllers disappear.

Keywords: Fuzzy control; MIMO control; PI control; relay control; variable gain control.

1. Introduction

Multiple-input multiple-output (MIMO) fuzzy controllers are desirable in many situations. However, the number of existing MIMO fuzzy control applications is rather small. Notable applications include the flexible wing aircraft control developed by Chiu et al. [4] and the steam generating unit regulation developed by Ray [9]. Some theoretical results on the traditional fuzzy-relation-based MIMO fuzzy controllers have also been obtained. In particular, Gupta et al. [5] studied the structure of a MIMO fuzzy control system. Kiszka et al. [6] investigated a MIMO fuzzy controller under Godel's implication. Xu and Lu [10] and Xu [11] did research on linguistic decoupling control of fuzzy multivariable processes. For detailed information on the status of fuzzy control theory, the reader is referred to the recent survey by Lee [7]. Summarily speaking, fuzzy control theory lags greatly behind fuzzy control applications. Compared to the well-developed conventional (i.e., nonfuzzy) control theory, fuzzy control theory, especially MIMO, is still in an early stage of development.

To develop fuzzy control theory, an analytical framework is necessary. The core of the framework should include analysis of analytical structures of fuzzy controllers with respect to conventional control theory. The analytical structures can then serve as a platform on which many analysis and synthesis methods can be analytically developed by utilizing powerful conventional control theory. The analytical structures will enable us to explain how and why fuzzy controllers work and to understand distinctive characteristics of fuzzy controllers compared to conventional controllers like the linear PID controller.

As a first step, the analytical structures of the single-input single-output (SISO) fuzzy controllers

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have been revealed, in terms of PI controller or multilevel relay controller, by Ying et al. [12] and Ying [13, 14]. Limiting structures of some fuzzy controllers with linear control rules were studied by Buckley and Ying [1] and Buckley [2]. Recently, we have exposed general analytical and limiting structures of typical SISO fuzzy controllers [15]. We have also proved mathematically that general fuzzy systems are function approximators [18]. In other words, general fuzzy systems can approximate any continuous functions in compact domains to any degree of accuracy.

We have extended the previously developed methodology [14] to more complicated typical SISO fuzzy controllers with *nonlinear* control rules and to a two-input two-output fuzzy controller with 16 simple fuzzy control rules. We have proved that a typical SISO fuzzy controller with nonlinear control rules is the sum of a global nonlinear controller and a local nonlinear PI-like controller [13]. We have shown that the two-input two-output fuzzy controller is the sum of two nonlinear PI controllers with variable gains changing continuously with process outputs [16].

The analytical structures of the fuzzy controllers disclosed in [13–15] enabled us to develop a systematic design procedure, along with a set of formulas, which can be used effectively to compute most of the parameters of typical SISO fuzzy controllers [19]. The design procedure eliminates most of the trial-and-error effort required previously in fuzzy controller design. In [3], we used the well-known Small Gain Theorem in conventional control theory to derive sufficient conditions for stability of fuzzy control systems involving the simplest possible fuzzy controller which is actually a nonlinear PI controller with variable gains [12]. The development of the design procedure and the stability conditions clearly illustrates the importance of investigating the analytical structure of fuzzy controllers in relation to conventional controllers. Furthermore, it demonstrates how fuzzy control problems can effectively be solved by the well-developed and powerful conventional control theory. There exist many powerful analytical analysis and design methods which can be borrowed to deal with many fundamental issues in fuzzy control such as stability, analysis, design and robustness, to name a few, provided that analytical structures of fuzzy controllers are available, and are in the form relevant to conventional controllers.

In this paper, we will derive the analytical structure of a two input two-output fuzzy controller with linear fuzzy control rules in terms of PI controller and multilevel relay. Fuzzy controllers with linear control rules are of interest primarily for two reasons. First, linear control rules represent reasonably well the typical control rules seen in fuzzy control literature [14, 19]. Second, linear control rules yield a relation to PI controller and multilevel relay in conventional control theory. Analytical structures of fuzzy controllers with nonlinear control rules are much more complicated even in SISO cases [13]. In the paper, the characteristics of the resultant analytical structure will quantitatively be investigated. The limiting structure of the fuzzy controller will also be derived.

2. Configuration of a two-input two-output fuzzy controller

Assume a two-input two-output process with coupling outputs, $y_1(k)$ and $y_2(k)$ (k is a positive integer representing sampling time), is controlled by a two-input two-output fuzzy controller to maintain the outputs around respective setpoints SP_1 and SP_2. Process output errors, denoted as $e_1(k)$ and $e_2(k)$, and rate change of errors (rates, for short), denoted as $\Delta e_1(k)$ and $\Delta e_2(k)$, are used as inputs of the fuzzy controller. The scaled inputs are

$$x_1(k) = k_1 \cdot e_1(k) = k_1(\mathrm{SP}_1 - y_1(k)), \tag{2.1}$$

$$x_2(k) = k_2 \cdot \Delta e_1(k) = k_2(y_1(k-1) - y_1(k)), \tag{2.2}$$

$$x_3(k) = k_3 \cdot e_2(k) = k_3(\mathrm{SP}_2 - y_2(k)), \tag{2.3}$$

$$x_4(k) = k_4 \cdot \Delta e_2(k) = k_4(y_2(k-1) - y_2(k)), \tag{2.4}$$

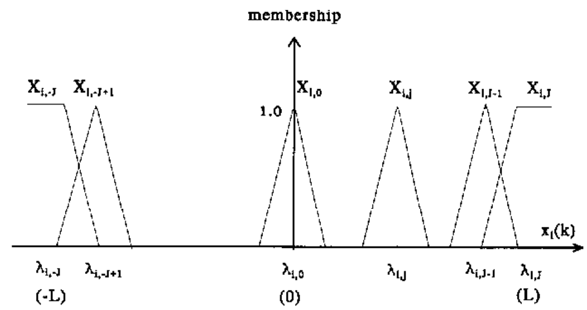

Fig. 1. Illustrative definition of the triangular-shaped membership functions of the input fuzzy sets. Note that central value of $X_{i,j}$ is $\lambda_{i,j}$ and $\lambda_{i,-J}=-L$, $\lambda_{i,0}=0$ and $\lambda_{i,J}=L$. Also, note that $\lambda_{i,j}-\lambda_{i,j-1}=S$.

where the k_i's ($i = 1, 2, 3, 4$) are input scalars and $k-1$ represents previous sampling time. We assume that the k_i's are chosen such that

$$-L \leqslant x_i(k) \leqslant L. \tag{2.5}$$

Each of the $x_i(k)$'s is fuzzified by the same input fuzzy set which has J ($J \geqslant 1$) members for positive $x_i(k)$, J members for negative $x_i(k)$ and one member for near zero $x_i(k)$. Therefore, the total number of members for $x_i(k)$ is

$$N = 2J + 1. \tag{2.6}$$

The members are represented by the following index system:

$$\{X_{i,-J}, X_{i,-J+1}, \ldots, X_{i,-1}, X_{i,0}, X_{i,1}, \ldots, X_{i,J-1}, X_{i,J}\}. \tag{2.7}$$

Membership functions are triangular-shaped as shown graphically in Figure 1. The membership functions corresponding to (2.7) are expressed as

$$\{\mu_{-J}(x_i), \mu_{-J+1}(x_i), \ldots, \mu_{-1}(x_i), \mu_0(x_i), \mu_1(x_i), \ldots, \mu_{J-1}(x_i), \mu_J(x_i)\}. \tag{2.8}$$

The central value of $X_{i,j}$ is denoted as $\lambda_{i,j}$ and $\lambda_{i,-J}=-L$, $\lambda_{i,0}=0$ and $\lambda_{i,J}=L$ (see Figure 1). Also, let the space between two adjacent members be equal. The space is

$$S = \frac{L}{J} \tag{2.9}$$

and therefore $\lambda_{i,j} = j \cdot S$.

In order to cover all possible combinations of the inputs, N^4 control rules are needed. The control rules used in this study must satisfy the following law:

$$\text{IF } x_1(k) = X_{1,j_1} \text{ AND } x_2(k) = X_{2,j_2} \text{ AND } x_3(k) = X_{3,j_3} \text{ AND } X_4(K) = X_{4,j_4}$$
$$\text{THEN } \Delta U_1(K) = \Delta U_{1,r_1}, \Delta u_2(k) = \Delta U_{2,r_2} \tag{2.10}$$

where

$$r_1 = j_1 + j_2 + a_1(j_3 + j_4), \tag{2.11}$$

$$r_2 = a_2(j_1 + j_2) + j_3 + j_4. \tag{2.12}$$

Control rule (2.10) is called a linear control rule because the linear functions are used in (2.11) and (2.12) to relate the indexes of the input fuzzy sets, j_1, j_2, j_3 and j_4, to the indexes of the output fuzzy sets, r_1 and r_2 (note that j_1, j_2, j_3, j_4, r_1 and r_2 are integers). The a_1 and a_2 are integers whose sign and magnitude depend upon how the process outputs are coupled. If increase of $y_1(k)$ causes increase of $y_2(k)$, a_2 should be negative. If increase of $y_1(k)$ causes decrease of $y_2(k)$, a_2 should be positive. Similarly, if increase of $y_2(k)$ causes increase of $y_1(k)$, a_1 should be negative. If increase of $y_2(k)$ causes decrease of $y_1(k)$, a_1 should be positive. When there is no coupling between the outputs, a_1 and a_2 should be zero. Further, the stronger the coupling between the process outputs, the larger the absolute value of a_1 and a_2 should be. In the control rule (2.10), the $\Delta u_m(k)$'s ($m = 1, 2$) are unscaled incremental outputs (outputs, for short) of the fuzzy controller, which satisfy

$$-H \leq \Delta u_m(k) \leq H. \tag{2.13}$$

There are two identical output fuzzy sets for the $\Delta u_m(k)$'s. Each output fuzzy set has $4J(1+a)+1$ members where

$$a = \mathrm{Max}\{|a_1|, |a_2|\}. \tag{2.14}$$

Among these members, $2J(1+a)$ members are for positive $\Delta u_m(k)$'s, $2J(1+a)$ members are for negative $\Delta u_m(k)$'s and one member is for near zero $\Delta u_m(k)$'s. The n-th member for $\Delta u_m(k)$ is represented by $\Delta U_{m,n}$ and all the members are described by

$$\{\Delta U_{m,-2J(1+a)}, \Delta U_{m,-2J(1+a)+1}, \ldots, \Delta U_{m,-1}, \Delta U_{m,0}, \Delta U_{m,1}, \ldots, \Delta U_{m,2J(1+a)-1}, \Delta U_{m,2J(1+a)}\}. \tag{2.15}$$

The membership function of $\Delta U_{m,n}$ is denoted as $\mu_n(\Delta u_m)$. In this study, the $\mu_n(\Delta u_m)$'s are trapezoidal-shaped and are symmetrical about central values, as shown in Figure 2. The central value of $\Delta U_{m,n}$ is designated as $\gamma_{m,n}$ and $\gamma_{m,-2J(1+a)} = -H$, $\gamma_{m,0} = 0$ and $\gamma_{m,2J(1+a)} = H$. Let the space between two adjacent members be equal. The space is

$$V = \frac{H}{2J(1+a)} \tag{2.16}$$

and consequently $\gamma_{m,n} = n \cdot V$. To define the shape of the trapezoids, a parameter

$$\theta = \frac{A}{V} \leq 0.5 \tag{2.17}$$

is used to avoid overlay between upper sides of two adjacent output fuzzy sets. Note that $2A$ and $2V$ are the upper and lower sides of the trapezoids, respectively.

Probabilistic AND fuzzy logic is employed to evaluate the AND's in the antecedents of the control rule (2.10). The resultant memberships are assigned to the consequences of the control rule, the $\Delta U_{m,r_m}$'s:

$$\mu_{r_m}(\Delta u_m) = \mu_{j_1}(x_1) \cdot \mu_{j_2}(x_2) \cdot \mu_{j_3}(x_3) \cdot \mu_{j_4}(x_4). \tag{2.18}$$

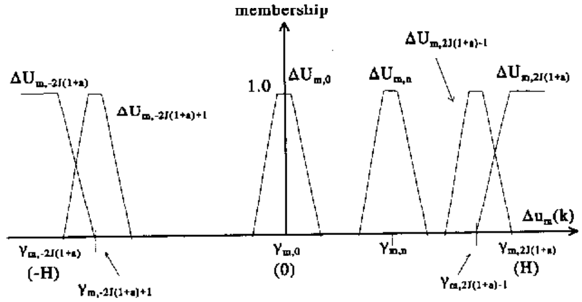

Fig. 2. Illustrative definition of the trapezoidal-shaped membership functions of the output fuzzy sets. Note that central value of $\Delta U_{m,n}$ is $\gamma_{m,n}$ and $\gamma_{m,-2J(1+a)} = -H$, $\gamma_{m,0} = 0$ and $\gamma_{m,2J(1+a)} = H$. Also, note that $\gamma_{m,n} - \gamma_{m,n-1} = V$.

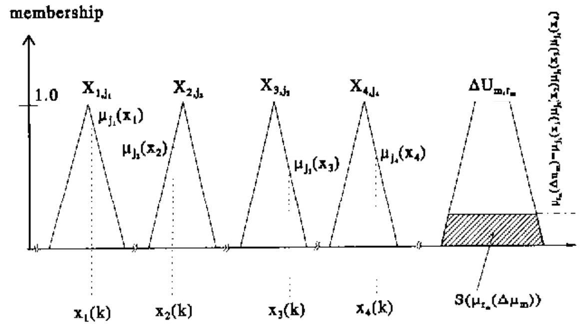

Fig. 3. Illustrative definition of Mamdani's minimum inference. Probabilistic AND fuzzy logic is used to calculate the membership of $\Delta U_{m,r_m}$ ($m = 1, 2$) in the linear control rule (2.10). That is, $\mu_{r_m}(\Delta u_m) = \mu_{j_1}(x_1)\mu_{j_2}(x_2)\mu_{j_3}(x_3)\mu_{j_4}(x_4)$. The shaded area can be calculated using formula (2.20). Note that $2A$ and $2V$ are the upper-side and lower-side of the trapezoidal-shaped $\Delta U_{m,r_m}$, respectively. Also note that the fuzzy sets X_{1,j_1}, X_{2,j_2}, X_{3,j_3}, X_{4,j_4} and $\Delta U_{m,r_m}$ are in five different universes of discourse.

If more than one control rule produces memberships for the same member, the Łukasiewicz OR fuzzy logic

$$F_1 \text{ OR } F_2 \text{ OR} \cdots \text{OR } F_w = \text{Min}\left(\sum_{p=1}^{w} \varphi_p, 1\right) \tag{2.19}$$

is used to yield a combined membership. The φ_p's are the memberships of the fuzzy sets F_p's.

To reason $\Delta U_{m,r_m}$'s from the antecedents of the control rules, Mamdani's minimum inference method, shown graphically in Figure 3, is employed [8]. The shaded area in Figure 3 is

$$S(\mu_{r_m}(\Delta u_m)) = (2 - \mu_{r_m}(\Delta u_m) + \theta \cdot \mu_{r_m}(\Delta u_m))\mu_{r_m}(\Delta u_m) \cdot V. \tag{2.20}$$

It should be pointed out that Figure 3 gives an illustrative representation of Mamdani's minimum inference method. The fuzzy sets X_{1,j_1}, X_{2,j_2}, X_{3,j_3}, X_{4,j_4} and $\Delta U_{m,r_m}$ in the figure are actually in five different universes of discourse.

The popular center of gravity defuzzification algorithm is utilized to attain $\Delta u_m(k)$'s from $\Delta U_{m,r_m}$'s. Since the shapes of the membership functions of $\Delta U_{m,r_m}$'s are identical and each membership function is symmetrical about its central value, the global centroids can be calculated from the local centroids which are the central values of the $\Delta U_{m,r_m}$'s involved. Hence, the scaled incremental outputs are

$$\alpha_m \cdot \Delta u_m(k) = \alpha_m \frac{\sum_{u_{r_m}(\Delta u_m)\neq 0} S(\mu_{r_m}(\Delta u_m)) \cdot \gamma_{m,r_m}}{\sum_{\mu_{r_m}(\Delta u_m)\neq 0} S(\mu_{r_m}(\Delta u_m))}, \quad m = 1, 2, \tag{2.21}$$

where the α_m's are output scalars. New outputs of the fuzzy controller are

$$u_m(k) = u_m(k-1) + \alpha_m \cdot \Delta u_m(k). \tag{2.22}$$

3. Analysis of structure of the fuzzy controller

3.1. Derivation of the analytical structure of the fuzzy controller

Definition 1 (see also [14]). A controller is called a global controller if its output is computed according to its inputs with respect to a fixed point in input space. In contrast, a controller is called a local controller if its output is calculated based on its inputs with respect to a moving point in input space.

These definitions will be used in Theorem 1. Also, without losing generality, we assume

$$j_i S \leq x_i(k) \leq (j_i + 1)S, \quad i = 1, 2, 3, 4, \tag{3.1}$$

where

$$-J \leqslant j_i \leqslant J-1. \tag{3.2}$$

We also denote

$$\Delta x_i = x_i(k) - (j_i + 0.5)S. \tag{3.3}$$

With these preparations, we state the main result of this paper as follows.

Theorem 1. *The analytical structure of the two-input two-output fuzzy controller is the sum of global four-dimensional multilevel relays, denoted as* $\Phi_m^{\mathrm{G}}(j_1, j_2, j_3, j_4)$, *and local nonlinear* PI *controllers with variable gains, denoted as* $\Phi_m^{\mathrm{L}}(\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4)$:

$$\alpha_m \cdot \Delta u_m(k) = \Phi_m^{\mathrm{G}}(j_1, j_2, j_3, j_4) + \Phi_m^{\mathrm{L}}(\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4), \quad m = 1, 2, \tag{3.4}$$

where

$$\Phi_1^{\mathrm{G}}(j_1, j_2, j_3, j_4) = (j_1 + j_2 + 1 + a_1(j_3 + j_4 + 1))\alpha_1 V, \tag{3.5}$$

$$\Phi_2^{\mathrm{G}}(j_1, j_2, j_3, j_4) = (a_2(j_1 + j_2 + 1) + j_3 + j_4 + 1)\alpha_2 V, \tag{3.6}$$

$$\begin{aligned}\Phi_1^{\mathrm{L}}(\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4) &= \frac{K_{\mathrm{i}}^1}{k_1}\Delta x_1 + \frac{K_{\mathrm{p}}^1}{k_2}\Delta x_2 + \delta_1\left(\frac{K_{\mathrm{i}}^2}{k_3}\Delta x_3 + \frac{K_{\mathrm{p}}^2}{k_4}\Delta x_4\right)\\ &= K_{\mathrm{i}}^1\left(e_1(k) - \frac{(j_1+0.5)S}{k_1}\right) + K_{\mathrm{p}}^1\left(\Delta e_1(k) - \frac{(j_2+0.5)S}{k_2}\right)\\ &\quad + \delta_1\left[K_{\mathrm{i}}^2\left(e_2(k) - \frac{(j_3+0.5)S}{k_3}\right) + K_{\mathrm{p}}^2\left(\Delta e_2(k) - \frac{(j_4+0.5)S}{k_4}\right)\right]\end{aligned} \tag{3.7}$$

$$\begin{aligned}\Phi_2^{\mathrm{L}}(\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4) &= \delta_2\left(\frac{K_{\mathrm{i}}^3}{k_1}\Delta x_1 + \frac{K_{\mathrm{p}}^3}{k_2}\Delta x_2\right) + \frac{K_{\mathrm{i}}^4}{k_3}\Delta x_3 + \frac{K_{\mathrm{p}}^4}{k_4}\Delta x_4\\ &= \delta_2\left[K_{\mathrm{i}}^3\left(e_1(K) - \frac{(j_1+0.5)S}{k_1}\right) + K_{\mathrm{p}}^3\left(\Delta e_1(k) - \frac{(j_2+0.5)S}{k_2}\right)\right]\\ &\quad + K_{\mathrm{i}}^4\left(e_2(K) - \frac{(j_3+0.5)S}{k_3}\right) + K_{\mathrm{p}}^4\left(\Delta e_2(k) - \frac{(j_4+0.5)S}{k_4}\right).\end{aligned} \tag{3.8}$$

The variable proportional-gains of the PI *controllers are*

$$K_{\mathrm{p}}^1 = \frac{\alpha_1\beta_1 k_2 H}{2L(1+a)}, \qquad K_{\mathrm{p}}^2 = \frac{\alpha_1\beta_2 k_4 H|a_1|}{2L(1+a)}, \qquad K_{\mathrm{p}}^3 = \frac{\alpha_2\beta_1 k_2 H|a_2|}{2L(1+a)}, \qquad K_{\mathrm{p}}^4 = \frac{\alpha_2\beta_2 k_4 H}{2L(1+a)}, \tag{3.9}$$

while the variable integral-gains are

$$K_{\mathrm{i}}^1 = \frac{\alpha_1\beta_1 k_1 H}{2L(1+a)}, \qquad K_{\mathrm{i}}^2 = \frac{\alpha_1\beta_2 k_3 H|a_1|}{2L(1+a)}, \qquad K_{\mathrm{i}}^3 = \frac{\alpha_2\beta_1 k_1 H|a_2|}{2L(1+a)}, \qquad K_{\mathrm{i}}^4 = \frac{\alpha_2\beta_2 k_3 H}{2L(1+a)}. \tag{3.10}$$

The β_1 *and* β_2 *are given in* (3.15) *and* (3.16), *respectively. The* δ_1 *is a unity with the same sign as* a_1 *while* δ_2 *is a unity with the same sign as* a_2.

Proof. As a result of fuzzifying $x_i(k)$, two memberships

$$\mu_{j_i}(x_i) \quad \text{and} \quad \mu_{j_i+1}(x_i) \tag{3.11}$$

are generated for the members X_{i,j_i} and X_{i,j_i+1}, respectively. More specifically,

$$\mu_{j_i}(x_i) = -\frac{1}{S}(\Delta x_i - 0.5S), \qquad \mu_{j_i+1}(x_i) = \frac{1}{S}(\Delta x_i + 0.5S). \tag{3.12}$$

Table 1. A list of the 16 activated control rules. The antecedents are X_{i,h_i} ($i = 1, 2, 3, 4$) and the consequences are $\Delta U_{m,r_m}$ ($m = 1, 2$)

IF $x_1(k) = X_{1,h_1}$ AND $x_2(k) = X_{2,h_2}$ AND $x_3(k) = X_{3,h_3}$ AND $x_4(k) = X_{4,h_4}$
THEN $\Delta u_1(k) = \Delta U_{1,r_1}$, $\Delta u_2(k) = \Delta U_{2,r_2}$

Rule No.	h_1	h_2	h_3	h_4	$r_1 = h_1 + h_2 + a_1(h_3 + h_4)$	$r_2 = a_2(h_1 + h_2) + h_3 + h_4$
R_1	j_1+1	j_2+1	j_3+1	j_4+1	$j_1+j_2+2+a_1(j_3+j_4+2)$	$a_2(j_1+j_2+2)+j_3+j_4+2$
R_2	j_1+1	j_2	j_3+1	j_4+1	$j_1+j_2+1+a_1(j_3+j_4+2)$	$a_2(j_1+j_2+1)+j_3+j_4+2$
R_3	j_1	j_2+1	j_3+1	j_4+1	$j_1+j_2+1+a_1(j_3+j_4+2)$	$a_2(j_1+j_2+1)+j_3+j_4+2$
R_4	j_1	j_2	j_3+1	j_4+1	$j_1+j_2+a_1(j_3+j_4+2)$	$a_2(j_1+j_2)+j_3+j_4+2$
R_5	j_1+1	j_2+1	j_3	j_4	$j_1+j_2+2+a_1(j_3+j_4)$	$a_2(j_1+j_2+2)+j_3+j_4$
R_6	j_1+1	j_2	j_3	j_4	$j_1+j_2+1+a_1(j_3+j_4)$	$a_2(j_1+j_2+1)+j_3+j_4$
R_7	j_1	j_2+1	j_3	j_4	$j_1+j_2+1+a_1(j_3+j_4)$	$a_2(j_1+j_2+1)+j_3+j_4$
R_8	j_1	j_2	j_3	j_4	$j_1+j_2+a_1(j_3+j_4)$	$a_2(j_1+j_2)+j_3+j_4$
R_9	j_1+1	j_2+1	j_3+1	j_4	$j_1+j_2+2+a_1(j_3+j_4+1)$	$a_2(j_1+j_2+2)+j_3+j_4+1$
R_{10}	j_1+1	j_2	j_3+1	j_4	$j_1+j_2+1+a_1(j_3+j_4+1)$	$a_2(j_1+j_2+1)+j_3+j_4+1$
R_{11}	j_1	j_2+1	j_3+1	j_4	$j_1+j_2+1+a_1(j_3+j_4+1)$	$a_2(j_1+j_2+1)+j_3+j_4+1$
R_{12}	j_1	j_2	j_3+1	j_4	$j_1+j_2+a_1(j_3+j_4+1)$	$a_2(j_1+j_2)+j_3+j_4+1$
R_{13}	j_1+1	j_2+1	j_3	j_4+1	$j_1+j_2+2+a_1(j_3+j_4+1)$	$a_2(j_1+j_2+2)+j_3+j_4+1$
R_{14}	j_1+1	j_2	j_3	j_4+1	$j_1+j_2+1+a_1(j_3+j_4+1)$	$a_2(j_1+j_2+1)+j_3+j_4+1$
R_{15}	j_1	j_2+1	j_3	j_4+1	$j_1+j_2+1+a_1(j_3+j_4+1)$	$a_2(j_1+j_2+1)+j_3+j_4+1$
R_{16}	j_1	j_2	j_3	j_4+1	$j_1+j_2+a_1(j_3+j_4+1)$	$a_2(j_1+j_2)+j_3+j_4+1$

A total number of $2^4 = 16$ control rules are executed. Table 1 lists all these 16 control rules and their consequences, the $\Delta U_{m,r_m}$'s. According to Table 1, some control rules produce memberships for the same members. These control rules can be grouped, for both $\Delta U_{1,r_1}$ and $\Delta U_{2,r_2}$, as follows: (1) R_2 and R_3; (2) R_6 and R_7; (3) R_9 and R_{13}; (4) R_{10}, R_{11}, R_{14} and R_{15}; and (5) R_{12} and R_{16}. Obviously, Łukasiewicz OR fuzzy logic (2.19) is needed to generate the combined memberships. It can be proved that the combined memberships equal to the sum of the memberships being combined. For example, the antecedents in R_2 yields $\mu_{j_1+1}(x_1) \cdot \mu_{j_2}(x_2) \cdot \mu_{j_3+1}(x_3) \cdot \mu_{j_4+1}(x_4)$ while the antecedents in R_3 produces $\mu_{j_1}(x_1) \cdot \mu_{j_2+1}(x_2) \cdot \mu_{j_3+1}(x_3) \cdot \mu_{j_4+1}(x_4)$. Note

$$\mu_{j_1+1}(x_1) \cdot \mu_{j_2}(x_2) + \mu_{j_1}(x_1) \cdot \mu_{j_2+1}(x_2) = \frac{1}{2} - \frac{2\Delta x_1\, \Delta x_2}{S^2} \leq 1. \tag{3.13}$$

Therefore, for R_2 and R_3,

$$\begin{aligned} &\mu_{j_1+1}(x_1) \cdot \mu_{j_2}(x_2) \cdot \mu_{j_3+1}(x_3) \cdot \mu_{j_4+1}(x_4) + \mu_{j_1}(x_1) \cdot \mu_{j_2+1}(x_2) \cdot \mu_{j_3+1}(x_3) \cdot \mu_{j_4+1}(x_4) \\ &\quad = (\mu_{j_1+1}(x_1) \cdot \mu_{j_2}(x_2) + \mu_{j_1}(x_1) \cdot \mu_{j_2+1}(x_2))\mu_{j_3+1}(x_3) \cdot \mu_{j_4+1}(x_4) \\ &\quad \leq \mu_{j_3+1}(x_3) \cdot \mu_{j_4+1}(x_4) \leq 1. \end{aligned} \tag{3.14}$$

Substitute the $\mu_{r_m}(\Delta u_m)$'s generated by the control rules R_1 to R_{16} into (2.20) to obtain $S(\mu_{r_m}(\Delta u_m))$'s. Then, put the $S(\mu_{r_m}(\Delta u_m))$'s into the defuzzification algorithm (2.21) and simplify the results. The structure of the fuzzy controller can be obtained as shown in (3.4). The β_1 and β_2 in (3.9) and (3.10) are

$$\beta_1 = \frac{N_1(\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4)}{D(\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4)}, \tag{3.15}$$

$$\beta_2 = \frac{N_2(\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4)}{D(\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4)} \tag{3.16}$$

where

$$N_1(\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4) = 2S^8 - S^2(1-\theta)(0.5S^4 + 2\Delta x_3^2 S^2 + 8\Delta x_3^2 \Delta x_4^2 + 2\Delta x_4^2)(\Delta x_1 \Delta x_2 + 0.25S^2), \tag{3.17}$$

$$N_2(\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4) = 2S^8 - S^2(1-\theta)(0.5S^4 + 2\Delta x_1^2 S^2 + 8\Delta x_1^2 \Delta x_2^2 + 2\Delta x_2^2)(\Delta x_3 \Delta x_4 + 0.25S^2), \tag{3.18}$$

$$\begin{aligned} D(\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4) &= (1.9375 + 0.0625\theta)S^8 - (1-\theta)[0.25S^2(\Delta x_1^2 + \Delta x_2^2)(S^4 + 2\Delta x_3^2 S^2 + 16\Delta x_3^2 \Delta x_4^2 + 2\Delta x_4^2 S^2) \\ &\quad + 0.25S^2(\Delta x_3^2 + \Delta x_4^2)(S^4 + 2\Delta x_1^2 S^2 + 16\Delta x_1^2 \Delta x_2^2 + 2\Delta x_2^2 S^2) \\ &\quad + (\Delta x_1^2 \Delta x_2^2 + \Delta x_3^2 \Delta x_4^2)S^4 + 16(\Delta x_1 \Delta x_2 \Delta x_3 \Delta x_4)^2]. \quad \square \end{aligned} \tag{3.19}$$

In Theorem 1, $\Phi_m^G(j_1, j_2, j_3, j_4)$'s are apparently two global four-dimensional multilevel relays with integer inputs, j_1, j_2, j_3 and j_4. The relays are global because

$$\begin{aligned} \Phi_1^G &= (j_1 + j_2 + 1 + a_1(j_3 + j_4 + 1))\alpha_1 V \\ &= \{(j_1 + 0.5)S + (j_2 + 0.5)S + a_1[(j_3 + 0.5)S + (j_4 + 0.5)S]\}\frac{\alpha_1 V}{S}, \end{aligned} \tag{3.20}$$

$$\begin{aligned} \Phi_2^G &= (a_2(j_1 + j_2 + 1) + j_3 + j_4 + 1)\alpha_2 V \\ &= \{a_2[(j_1 + 0.5)S + (j_2 + 0.5)S] + (j_3 + 0.5)S + (j_4 + 0.5)S\}\frac{\alpha_2 V}{S}, \end{aligned} \tag{3.21}$$

which means that the control actions of the relays are computed according to the middle points of the intervals $[j_i S, (j_i + 1)S]$ ($i = 1, 2, 3, 4$), where the $x_i(k)$'s lie, with respect to the origin of the scaled input space (that is, the origin is the fixed point in Definition 1). It is important to note that the control actions of the global relays remain unchanged in the situations that the x_i's change but not enough to cause change of the j_i's (i.e., the x_i's remain in the same intervals $[j_i S, (j_i + 1)S]$, for $i = 1, 2, 3, 4$).

The $\Phi_m^L(\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4)$'s in Theorem 1 are local nonlinear PI controllers with variable proportional-gains and integral-gains and with local and changing steady-states, $(j_i + 0.5)S/k_i$. The PI controllers are local because their control actions are calculated based on Δx_i's, as shown in (3.7) and (3.8), which are the distances between $x_i(k)$ and $(j_i + 0.5)S$, the middle points of the intervals $[j_i S, (j_i + 1)S]$ (the moving point in Definition 1). Unlike the global multilevel relays, the control actions of the local nonlinear PI controllers always change with change of the x_i's.

In essence, Theorem 1 states that the fuzzy controller outputs consist of two parts: the first part is generated by the global multilevel relays and the second part by the nonlinear PI controllers with variable proportional-gains and integral-gains. The coupling between the process outputs, described by the parameters a_1 and a_2, affects the multilevel relays and the nonlinear PI controllers. The resulting expressions in Theorem 1 should be viewed in the context of conventional control theory rather than purely mathematical expressions. The expressions are as compact as possible, although they still look quite complicated. The complexity of some expressions in Theorem 1 such as those in (3.15) to (3.19) demonstrates that fuzzy controllers are nonlinear controllers with inherent and complex nonlinearity. The complexity also implies the technical difficulty of obtaining the results and suggests how difficult it could be to find out analytical structure of other MIMO fuzzy controllers, especially those with nonlinear fuzzy control rules.

In the following, we will analytically investigate some important properties of the fuzzy controller.

3.2. Properties of the fuzzy controller

Theorem 2. *The fuzzy controller has the following structural dualities (for $m = 1, 2$):*

$$\Phi_m^G(j_1, j_2, j_3, j_4) = \Phi_m^G(j_2, j_1, j_4, j_3), \tag{3.22}$$

$$\Phi_m^L(\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4) = \Phi_m^L(\Delta x_2, \Delta x_1, \Delta x_4, \Delta x_3). \tag{3.23}$$

Proof. The proof is omitted because it is straightforward. When proving (3.23), one should note that

$$N_1(\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4) = N_1(\Delta x_2, \Delta x_1, \Delta x_4, \Delta x_3), \tag{3.24}$$

$$N_2(\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4) = N_2(\Delta x_2, \Delta x_1, \Delta x_4, \Delta x_3), \tag{3.25}$$

$$D(\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4) = D(\Delta x_2, \Delta x_1, \Delta x_4, \Delta x_3). \quad \square \tag{3.26}$$

Theorem 2 indicates that if x_1 and x_2, and x_3 and x_4 are interchanged, the outputs of the fuzzy controller will remain the same. This property is largely due to the identity of the input fuzzy sets.

As pointed out earlier, the local controllers are nonlinear PI controllers with variable proportional-gains and integral-gains. The sole source causing the gains to vary is β_1 and β_2 in (3.9) and (3.10). According to (3.15) and (3.16), β_1 and β_2 change continuously with the controller inputs, x_1, x_2, x_3 and x_4. Hence, we need to study the properties of β_1 and β_2 in order to understand the local controllers. The followings are some results.

Theorem 3. *For* $p = 1, 2,$

(i) $\beta_p(\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4) = \beta_p(\Delta x_2, \Delta x_1, \Delta x_4, \Delta x_3),$ (3.27)

where the $\beta_p(\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4)$*'s represent the* β_p*'s in* (3.15) *and* (3.16), *respectively.*

(ii) $\beta_p^{\min} \le \beta_p \le \beta_p^{\max},$ (3.28)

where the maximal and minimal β_p*'s are*

$$\beta_p^{\max} = \frac{2}{1+\theta}, \tag{3.29}$$

$$\beta_p^{\min} = \frac{2(3+\theta)}{7+\theta}. \tag{3.30}$$

(iii) *The range of the* β_p*'s, described by the ratio* $\beta_p^{\max}/\beta_p^{\min}$, *decreases monotonically as* θ *increases, and*

$$1.429 \le \frac{\beta_p^{\max}}{\beta_p^{\min}} \le 2.333. \tag{3.31}$$

The maximal and minimal ratios take place when $\theta = 0$ *and* $\theta = 0.5$, *respectively.*

Proof. (i) The results are apparent because of the structural dualities described in (3.24) to (3.26).

(ii) The numerators of the β_p's achieve their maximum, $2S^8$, while the denominators attain their minimum, $(1+\theta)S^8$, when $\Delta x_1 \Delta x_2 = -0.25S^2$ ($\Delta x_1 = 0.5S$ and $\Delta x_2 = -0.5S$ or $\Delta x_1 = -0.5S$ and $\Delta x_2 = 0.5S$) and $|\Delta x_3| = |\Delta x_4| = 0.5S$. Hence, (3.29) is proved.

β_1 reaches its minimum (3.30) when $\Delta x_1 = \Delta x_2 = 0$ and $|\Delta x_3| = |\Delta x_4| = 0.5S$. Based on the structural duality of the β_p's described in (3.27), it is obvious that β_2 reaches its minimum (3.30) when $\Delta x_3 = \Delta x_4 = 0$ and $|\Delta x_1| = |\Delta x_2| = 0.5S$.

(iii) Note that

$$\frac{\beta_p^{\max}}{\beta_p^{\min}} = \frac{7+\theta}{(1+\theta)(3+\theta)} \tag{3.32}$$

which decreases monotonically as θ increases. The results in (3.31) follow. $\square$

Theorem 3 tells that x_1 and x_2, and x_3 and x_4 are interchangeable as far as β_1 and β_2 are concerned. It also shows that the maximum and minimum of β_1 and β_2 depend only on θ which determines the shape of the output fuzzy sets. The range of variation of β_1 and β_2 decreases monotonically as θ increases. The maximal range is achieved when $\theta = 0$ while the minimal range is obtained when

$\theta = 0.5$. Based on the ranges of the β_p's ($p = 1, 2$), the ranges of K_p^i's and K_I^i's in (3.9) and (3.10), which are the proportional-gains and integral-gains of the local nonlinear PI controllers, can be determined accordingly.

Recall that the outputs of the fuzzy controller consist of the contribution of the global multilevel relays and the local nonlinear PI controllers. In what follows, we will specify quantitatively the contribution of these controllers in total control outputs.

Theorem 4. (i) *The absolute value of the maximal control actions of the global multilevel relays is*

$$|\Phi_m^G(j_1, j_2, j_3, j_4)|_{\max} - \frac{N-2}{N-1} \cdot \frac{1+|a_m|}{1+a} \alpha_m H. \tag{3.33}$$

(ii) *The absolute value of the maximal control actions of the local nonlinear* PI *controllers is*

$$|\Phi_m^L(\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4)|_{\max} = \frac{1}{N-1} \cdot \frac{1+|a_m|}{1+a} \alpha_m H. \tag{3.34}$$

(iii) *The absolute value of the maximal control actions of the fuzzy controller is*

$$\begin{aligned} |\alpha_m \cdot \Delta u_m(k)|_{\max} &= |\Phi_m^G(j_1, j_2, j_3, j_4) + \Phi_m^L(\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4)|_{\max} \\ &- |\Phi_m^G(j_1, j_2, j_3, j_4)|_{\max} + |\Phi_m^L(\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4)|_{\max} = \frac{1+|a_m|}{1+a} \alpha_m H. \end{aligned} \tag{3.35}$$

Proof. (i) When $a_1 \geqslant 0$,

$$\begin{aligned} |\Phi_1^G(j_1, j_2, j_3, j_4)|_{\max} &= \Phi_1^G(J-1, J-1, J-1, J-1) \\ &= (2J - 1 + a_1(2J-1))\alpha_1 V = \frac{N-2}{N-1} \cdot \frac{1+a_1}{1+a} \alpha_1 H. \end{aligned} \tag{3.36}$$

When $a_1 < 0$,

$$\begin{aligned} |\Phi_1^G(j_1, j_2, j_3, j_4)|_{\max} &= \Phi_1^G(J-1, J-1, -J, -J) \\ &= (2J - 1 + |a_1|(2J-1))\alpha_1 V = \frac{N-2}{N-1} \cdot \frac{1+|a_1|}{1+a} \alpha_1 H. \end{aligned} \tag{3.37}$$

Similarly, $|\Phi_2^G(j_1, j_2, j_3, j_4)|_{\max}$ can be proven.

(ii) When $a_1 \geqslant 0$,

$$|\Phi_1^L(\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4)|_{\max} = \Phi_1^L(0.5S, 0.5S, 0.5S, 0.5S) = \frac{1}{N-1} \cdot \frac{1+a_1}{1+a} \alpha_1 H. \tag{3.38}$$

When $a_1 < 0$,

$$|\Phi_1^L(\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4)|_{\max} = \Phi_1^L(0.5S, 0.5S, -0.5S, -0.5S) = \frac{1}{N-1} \cdot \frac{1+|a_1|}{1+a} \alpha_1 H. \tag{3.39}$$

By the same token, $|\Phi_2^L(\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4)|_{\max}$ can also be proven.

(iii) The proof is obvious based on the proof of (3.33) and (3.34). □

On the basis of Theorem 4, the conclusion follows that the control actions of the global multilevel relays are $N-2$ times larger than those of the local nonlinear PI controllers. To quantitatively represent the role of the relays and the PI controllers in the total control actions of the fuzzy controller, we introduce (see also [14])

$$\rho = \frac{|\Phi_m^L(\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4)|_{\max}}{|\alpha_m \cdot \Delta u_m(k)|_{\max}} \times 100\% = \frac{1}{N-1} \times 100\%. \tag{3.40}$$

When $N=3$, which is a minimal N, ρ achieves its maximum, 50%, meaning the global multilevel relays and the local nonlinear PI controllers play an equally important role in the total control actions of the fuzzy controller. When $N>3$, $\rho<50\%$, which indicates the relays play a major role in determining control actions of the fuzzy controller. What will happen if $N\to\infty$ ($\rho\to 0$)? The following theorem reveals the limiting structure of the fuzzy controller.

Theorem 5 (Limiting Structure Theorem). *When* $N\to\infty$:

(i) *The local nonlinear* PI *controllers disappear. That is,*

$$\lim_{N\to\infty}\Phi_m^{\mathrm{L}}(\Delta x_1,\Delta x_2,\Delta x_3,\Delta x_4)=0,\quad m=1,2. \tag{3.41}$$

(ii) *The limiting structure of the fuzzy controller is the sum of the linear* PI *controllers*:

$$\begin{aligned}\lim_{N\to\infty}\alpha_1\cdot\Delta u_1(k)&=\lim_{N\to\infty}\Phi_1^{\mathrm{G}}(j_1,j_2,j_3,j_4)\\&=\bar{K}_\mathrm{i}^1\cdot e_1(nT)+\bar{K}_\mathrm{p}^1\cdot\Delta e_1(nT)+\delta_1(\bar{K}_\mathrm{i}^2\cdot e_2(nT)+\bar{K}_\mathrm{p}^2\cdot\Delta e_2(nT)),\end{aligned} \tag{3.42}$$

$$\begin{aligned}\lim_{N\to\infty}\alpha_2\cdot\Delta u_2(k)&=\lim_{N\to\infty}\Delta_2^{\mathrm{G}}(j_1,j_2,j_3,j_4)\\&=\delta_2(\bar{K}_\mathrm{i}^3\cdot e_1(nT)+\bar{K}_\mathrm{p}^3\cdot\Delta e_1(nT))+\bar{K}_\mathrm{i}^4\cdot e_2(nT)+\bar{K}_\mathrm{p}^4\cdot\Delta e_2(nT),\end{aligned} \tag{3.43}$$

where

$$\bar{K}_\mathrm{p}^1=\frac{\alpha_1k_2H}{2L(1+a)},\quad \bar{K}_\mathrm{p}^2-\frac{\alpha_1k_4H|a_1|}{2L(1+a)},\quad \bar{K}_\mathrm{p}^3=\frac{\alpha_2k_2H|a_2|}{2L(1+a)},\quad \bar{K}_\mathrm{p}^4=\frac{\alpha_2k_4H}{2L(1+a)}, \tag{3.44}$$

$$\bar{K}_i^1=\frac{\alpha_1k_1H}{2L(1+a)},\quad \bar{K}_\mathrm{i}^2=\frac{\alpha_1k_3H|a_1|}{2L(1+a)},\quad \bar{K}_\mathrm{i}^3=\frac{\alpha_2k_1H|a_2|}{2L(1+a)},\quad \bar{K}_\mathrm{i}^4=\frac{\alpha_2k_3H}{2L(1+a)}. \tag{3.45}$$

Proof. (i)

$$\begin{aligned}\lim_{N\to\infty}\Phi_m^{\mathrm{L}}(\Delta x_1,\Delta x_2,\Delta x_3,\Delta x_4)&\leqslant\lim_{N\to\infty}|\Delta_m^{\mathrm{L}}(\Delta x_1,\Delta x_2,\Delta x_3,\Delta x_4)|_{\max}\\&=\lim_{N\to\infty}\frac{1}{N-1}\cdot\frac{1+|a_m|}{1+a}\alpha_mH=0,\end{aligned} \tag{3.46}$$

(ii) According to (3.1),

$$\frac{j_i}{J}\leqslant\frac{x_i(k)}{L}\leqslant\frac{j_i+1}{J},\quad i=1,2,3,4. \tag{3.47}$$

Therefore,

$$\lim_{N\to\infty}\frac{j_i}{L}=\frac{x_i(k)}{L}. \tag{3.48}$$

As a result,

$$\begin{aligned}\lim_{N\to\infty}\alpha_1\cdot\Delta u_1(k)&=\lim_{N\to\infty}\Phi_1^{\mathrm{G}}(j_1,j_2,j_3,j_4)=\lim_{N\to\infty}\left[\frac{j_1}{J}+\frac{j_2}{J}+a_1\left(\frac{j_3}{J}+\frac{j_4}{J}\right)\right]\frac{\alpha_1H}{2(1+a)}\\&=\bar{K}_\mathrm{i}^1\cdot e_1(nT)+\bar{K}_\mathrm{p}^1\cdot\Delta e_1(nT)+\delta_1(\bar{K}_\mathrm{i}^2\cdot e_2(nT)+K_\mathrm{p}^2\cdot\Delta e_2(nT))\end{aligned} \tag{3.49}$$

which are the sum of two global linear PI controllers. Obviously, the result of $\lim_{N\to\infty}\alpha_2\cdot\Delta u_2(k)$ in (3.43) can be proved in the same way. □

Theorem 5 not only exposes the limiting structure of the fuzzy controller but also indicates that ρ

represents linearity of the fuzzy controller. The smaller the ρ, the more linear the fuzzy controller; and vice versa. The global multilevel relays become the sum of the global linear PI controllers as $\rho = 0$.

The results in this section extend the results on the SISO fuzzy controller in [14] to the two-input two-output fuzzy controller in this paper. It should be pointed out that output fuzzy sets, fuzzy logic AND and fuzzy inference methods used in this paper are different from those in the SISO fuzzy controller in [14]. The analytical analysis approach is similar but technical difficulty is much greater for the fuzzy controller in this paper. The results in this paper and those in [14] are consistent in that when linear fuzzy control rules are used, the analytical structure of the fuzzy controllers can be related to global multilevel relays and local nonlinear PI controllers with variable proportional-gains and integral-gains. Also, the definition of the linearity of the fuzzy controllers (ρ) and the limiting structure of the fuzzy controllers agree in principle in these two cases.

The analysis in this section provides an analytical insight and precise understanding on how the fuzzy controller works. In general, the major portion of the controller outputs is from the global four-dimensional multilevel relays since in practice N is often greater than three. This control action is locally fine-tuned by the local nonlinear PI controllers with variable gains, which are in fact nonlinear adaptive PI controllers [17]. The larger the number of fuzzy control rules, the more contribution from the global multilevel relays and less contribution from the local nonlinear PI controllers, and the degree of nonlinearity decreases as well. As an extreme, as the number of control rules approaches ∞, each global multilevel relay becomes the sum of two global linear PI controllers while the local nonlinear PI controllers disappear. That is, the fuzzy controller becomes a linear controller.

4. Conclusions

It has been proved in the paper that the analytical structure of the two-input two-output fuzzy controller with linear control rules is the sum of global four-dimensional multilevel relays and local nonlinear PI controllers whose proportional-gains and integral-gains continuously change with process outputs. Various properties of the fuzzy controller have been investigated. The roles of the global multilevel relays and the local nonlinear PI controllers in the total control actions of the fuzzy controller have quantitatively been defined. It has been found that, except when $N = 3$, the global multilevel relays play a global and major role while the nonlinear PI controllers play a local and minor role in determining the control action of the fuzzy controller. It has also been proved that, as the number of control rules approaches ∞, the local nonlinear PI controllers disappear while the relays become the sum of two global linear PI controllers.

To our knowledge, the analytical structure in this paper and the one in [16] are the only two analytical structures available for multiple-input multiple-output fuzzy controllers. One of the important aspects of these analytical structures is that they are closely related to nonlinear PI controller and multilevel relay in conventional control theory. Such relations bridge the wide gap currently existing between fuzzy control theory and conventional control theory.

The results in this paper may be used to develop systematic design procedures for MIMO fuzzy controllers with nonlinear control rules, just like what we did with the results in [14] for the SISO fuzzy controllers [19]. Furthermore, the results may be extended to yield general analytical and limiting structures of general MIMO fuzzy controllers, obtaining results in parallel to those in [15, 18]. Also, various classic analysis methods, such as the Lyapunov method and the Small Gain Theorem in conventional control theory, may be employed to study ability of MIMO fuzzy control systems involving this fuzzy controller.

Based on the methodology developed in this paper, analytical structures of other MIMO fuzzy controllers with more inputs, more outputs or nonlinear fuzzy control rules can be derived. However, it is important to be aware that the more inputs and outputs are involved, the more complex the structures of the fuzzy controllers and the more difficult to get their analytical structures. The complexity increases exponentially with increase of the number of process outputs. For a MIMO fuzzy controller using errors and rates of κ process outputs as inputs, at least 4^{κ} control rules are activated at any given sampling time. Hence, when $\kappa > 3$, the difficulty of finding the analytical structure is conceivable.

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