Theory of Extended Fuzzy Discrete-Event Systems for Handling Ranges of Knowledge Uncertainties and Subjectivity

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Abstract—In 2001, we originated a theory of fuzzy discrete-event systems (FDESs) that generalized the conventional/crisp discrete-event systems (DESs). Vagueness and imprecision concerning states and event transitions of DESs were represented by membership grades and computed via fuzzy logic. Our application of the FDES theory to computerized human immunodeficiency virus/acquired immune deficiency syndrome treatment regimen selection, although preliminarily successful, suggests that a more comprehensive FDES theory is needed to address two general issues critically important not only to biomedical applications, but also to real-world problems in other industries. First, domain experts should have means other than point estimates and type-1 fuzzy sets mandated in the current framework to describe uncertainties, subjectivity, and imprecision in their (complex) knowledge and experience. Second, when a group of experts with distinct opinions is involved, they should not be forced to reach consensus for the sake of system development. This is because collective consensus may not be achievable, which is often the case in medicine, where individual experts’ opinions should be equally respected since the underlying ground truth is unknown most of the time. The theory of extended FDES presented in this paper addresses both the problems and contains the FDES theory as a special case. Experts are now allowed to use interval numbers and type-2 fuzzy sets to intuitively and quantitatively express their diverse knowledge and experience, which will then be processed by the new theory to form fuzzy state vectors and fuzzy event transition matrices. Accordingly, we have established mathematical operations that cover the computations of fuzzy states, fuzzy event transitions, and parallel composition. Numerical examples are provided.

Index Terms—Automata, discrete-event systems (DESs), fuzzy logic, type-2 fuzzy systems.

I. INTRODUCTION

There exist countless (complex) systems that cannot be effectively described, at a higher level, by differential equations, but can be by traces (or sequences) of events that record significant qualitative changes in the state of the system. These states are logical or symbolic rather than numerical. These systems can be described as discrete-event systems (DESs) whose behavior consists of sequence of occurrences of distinct events. These events, for instance, can describe sending or receiving messages in computer networks, or processing a part in a manufacturing plant. Comprehensive study and development of DES theory is a recent endeavor. Only after the proliferation of complex systems such as computer systems and networks, have DESs been systematically studied. DES theory addresses the issues of modeling, control, and optimization of DES [1]–[3]. It has been applied to practical systems such as automated manufacturing systems, database concurrency control, feature interactions in telecommunication networks, protocol verifications and synthesis in computer networks, and protocol conversion and gateway synthesis in computer networks. Although seemingly sufficient for many application domains, crisp DES theory is not adequate for some important fields such as biomedicine in which the state and state transition of a system (e.g., a person’s health status) are always somewhat uncertain and vague even in a deterministic sense. Subjective human observation, judgment, and interpretation (e.g., by a physician or a patient) invariably play a significant role in describing the status of state, usually not crisp.

Definitions of fuzzy state, fuzzy event, and fuzzy automata were proposed and studied as early as in the late 1960s [4], [5]. However, only a few studies followed in the next 30 years or so [6]–[8]. A comprehensive theory of fuzzy DESs (FDESs) was yet to be established, so were many fundamentally important concepts, methods, and theorems in the traditional DES, such as controllability, observability, and optimal control. To effectively represent deterministic uncertainties and vagueness as well as human subjective observation and judgment encountered in many real-world problems especially those in medicine, we recently originated a more comprehensive theory of FDES [9], [10]. We introduced fuzzy states and fuzzy event transition and generalized conventional/crisp DES to FDES. The largely graph-based framework of DES was unsuitable for the expansion, and we thus reformulated it using state vectors and event transition matrices that could be extended to fuzzy vectors and matrices by allowing their elements to take values in [0, 1]. We also extended optimal control and observability of DES to FDES. The new observability allows one to determine whether or not the system output observed is sufficient for decision making. The FDES theory is consistent with the DES theory, both at conceptual and computation levels, in that the former contains the latter as a special case when the membership grades are either 0 or 1. Other researchers have expanded different aspects of the FDES theory [11]–[20]. Other independent studies include [21] and [22]. More recently, we further developed the FDES theory so that it possessed self-learning capability [23].

As the first application, we have successfully applied the FDES theory to develop a novel system that prescribes drugs for human immunodeficiency virus (HIV)/acquired immune
deficiency syndrome (AIDS) patients who are about to receive their first combination antiretroviral treatment (the only effective long-term treatment to date) [24, 25]. The United Nations estimates that 38 million people worldwide are infected with HIV and that more than 22 million have died. HIV/AIDS treatment is, unfortunately, among one of the most complex treatments for any disease. Although the U.S. Department of Human Health and Services HIV/AIDS treatment guidelines cover the first-round combination antiretroviral therapy, they do not provide individualized treatment advice. In this regard, computer software that can utilize clinical information to recommend regimens (a regimen is a combination of different drugs) that will be sufficiently potent, well tolerated, and taken on schedule by an HIV-infected person would be a great advance. Accordingly, the objective of our project was to build such a system whose regimen choice for any given patient will match expert AIDS physician’s selection to produce the (anticipated) optimal treatment outcome. Preliminary retrospective evaluation of our prototype system using patients treated in our institution’s AIDS Clinical Center demonstrated encouraging results when the system operated in either self-learning mode or nonlearning mode.

Our first-hand application experience indicates that a more comprehensive FDES theory is needed to address two general issues critically important not only to biomedical applications, but also to applications in other industries. First, extracting application-specific knowledge and experience from domain experts is a crucial but challenging task for both the system developer and the domain expert. It is usually not easy for the expert to explicitly express his/her ideas qualitatively. Doing the same quantitatively is obviously even harder. From the system developer’s standpoint, how to accurately translate the domain expertise into an FDES-useable form is a major technical hurdle. The expert should have means other than point estimates and type-1 fuzzy sets mandated in the current framework to describe uncertainties, subjectivity, and imprecision in his/her knowledge and experience. Let us use our HIV/AIDS application as a concrete example to illustrate this issue.

A physician must consider many factors when selecting an antiretroviral treatment regimen for a patient. In our FDES scheme, the primary four clinical parameters germane to this decision are as follows: 1) anticipated potency of the regimen; 2) anticipated adherence for the patient under the regimen; 3) prognosis for adverse events under the regimen; and 4) expected future drug options due to the potential for development of resistance to the regimen. We used the literature-cited percentage of patients who achieved plasma HIV ribonucleic acid (RNA) less than 400 copies/mL after 48 weeks of treatment as our measure of potency. Adherence is a very complicated issue involving many factors impacting the patient’s ability to comply with the prescribed regimen. We defined adherence as expected percentage of drug doses prescribed by the physician that were actually taken by the patient weekly for each regimen. The parameter adverse events was defined as undesirable side effects and toxicities. Finally, the factor of future drug options meant drugs available after the current treatment was no longer viable. In the knowledge acquisition phase, one of the tasks of the two AIDS expert physicians (i.e., domain experts) in our team was to estimate, based on their experience and knowledge, a percentage number to each of the four factors for all the treatment regimens under consideration (e.g., potency is 85% for regimen X, adherence is 80% for regimen Y, and adverse events is 20% for regimen Z). This turned out to be a difficult task for the physicians. While the percentages were subjective summarization of the clinical possibilities for the parameters and represented the experts’ best point estimates, there were uncertainties as the true values did not exist in the literature, and consequently, the ground truth was (and still is) unknown. It is obvious that allowing the experts to use interval numbers (e.g., potency is [82%, 88%] for regimen X) or fuzzy numbers (e.g., potency is around 85%), impossible in the current FDES theory, would make the knowledge acquisition process much easier and more intuitive for the experts with more realistic knowledge representation. A fuzzy set can be considered as a fuzzy number if a set of conditions described later is met.

The second issue is that when a group of experts with distinct opinions is involved, they should not be forced to reach consensus for the sake of system development. This is because collective consensus may not be achievable, which is often the case in medicine, where individual experts’ opinions should be equally respected since the underlying ground truth is unknown most of the time. This can also be true for industrial applications. Continue the aforementioned HIV/AIDS treatment example. If physician A thinks potency of regimen X to be 83%, but physician B judges it to be 87%, under the present FDES framework, they are forced to compromise and/or reach a consensus (e.g., potency is 85%). The adoption of interval values or fuzzy numbers (e.g., potency is [82%, 88%] or around 85%) would better represent the diverse estimates.

A similar scenario occurs when the expert physicians defined fuzzy sets to linguistically describe different levels of the four clinical parameters. For instance, two fuzzy sets “high” and “medium,” as shown in Fig. 1, were chosen to characterize the possibilities of the system changing from the initial state (i.e.,
untreated mode) to “high” or “medium” state as far as potency is concerned. The fuzzy sets represented the consensus of the two experts. The knowledge acquisition could not have been accomplished had the experts failed to reach consensuses (different experts may insist to use different fuzzy sets to reflect their own experiences and knowledge). Clearly, a much better way is that experts will not be forced to reach consensus to begin with. An effective approach is to use a type-2 fuzzy set [26], which utilizes a lower primary membership function and an upper primary membership function to bound an infinite number of type-1 membership functions to characterize imprecision and uncertainties. In addition, a secondary membership function can be defined over the membership grades at each value of potency. These issues motivated us to expand the FDES theory so that they would be effectively dealt with. It turns out that all the elements in fuzzy state vectors and fuzzy event transition matrices must be fuzzy numbers, as opposed to crisp numbers under the current framework. The operations of the fuzzy numbers are based on interval number operations. To differentiate the new FDES from the existing ones, we call them the extended FDES (EFDES). We have also extended the theory to cover the parallel composition of EFDES. Like the transition from DES to FDES where the latter contains the former as a special case, FDES is a special case of EFDES because a crisp number is a special case of a fuzzy number. That means the theory developed in this paper can be applied to FDES as well.

II. THEORY OF EFDES

We first point out that multiple FDESs or EFDES running concurrently are required when the problem of interest is not very simple. This is the case for our HIV/AIDS application, for example. To make the presentation better in Section II-A and Section II-B1–II-B3, we will focus only on a single FDES or EFDES system when describing the theories. Before developing the EFDES theory, we will first provide a brief overview of the FDES theory. More detailed information on the theory is available in [10].

A. Outline of the FDESs Theory

A general FDES is modeled by a fuzzy automation

$$\hat{G} = (\hat{Q}, \hat{\Sigma}, \hat{\delta}, \hat{q}_0)$$

(1)

where $\hat{Q}$ is the set of all possible fuzzy state vectors and $\hat{q}_0 \in \hat{Q}$ is the initial state vector. The state vector $\hat{q}_k = [kV_1, kV_2, \ldots, kV_N] \in \hat{Q}$ with $N$ being the total number of states and $k = 1, 2, \ldots$, indicates time instance. Subscripts and superscripts are positive or nonnegative integers unless pointed out otherwise. The state variable $kV_i \in [0, 1]$ represents the possibility (i.e., membership) of the system being in state $i$. $\hat{\Sigma}$ is the set of all possible events. The $j$th event is represented by an event transition matrix $\hat{\sigma}_j = [a_{i,m}]_{N \times N}$ where the element $a_{i,m} \in [0, 1]$ describes the likelihood (i.e., membership) of the system moving from state $m$ to state $n$. $\hat{\delta} : \hat{Q} \circ \hat{\Sigma} \rightarrow \hat{Q}$ is a transition mapping that describes what event can occur at the current time and what the resulting new state is. Here, the symbol $\circ$ represents some fuzzy operation specified by $\delta$. It includes, but not limited to, the max–product or max–min fuzzy inference operation in the fuzzy set theory [27]. Which operation to use is application-dependent and is up to the system developer. Max–product or max–min is an operation in which the operators max($\cdot$), product($\cdot$), and min($\cdot$) are used. Product($\cdot$) means multiplication of the arguments while max($\cdot$) [or min($\cdot$)] operation picks up the largest (or smallest) argument among the arguments involved. If the current fuzzy state vector is $\hat{q}_k$ and the event $\hat{\sigma}_j$ occurs, then the new fuzzy state vector can be computed: $\hat{q}_{k+1} = \hat{q}_k \circ \hat{\sigma}_j$.

We also extended the parallel composition of DES to FDES [10]. In the interest of brevity, they will not be presented here. In the FDES framework, all the memberships in the fuzzy state vectors and fuzzy event transition matrices are crisp numbers in [0, 1]. While this type of membership representation is adequate for many applications, there are situations where it may not be powerful enough to handle complex uncertainties (recall the issues mentioned in Section I and we will demonstrate more specific cases later). To address this problem, we propose to use type-1 fuzzy sets to characterize the memberships, leading to the EFDES theory presented next. We will use the notations established in this section and expand them to present the new theory in a parallel fashion.

B. EFDESs Theory

1) Conceptual and Notational Extensions From the FDES Theory: Like FDES, a general EFDES is also modeled by a fuzzy automaton

$$G = (Q, \Sigma, \delta, q_0)$$

(2)

where $Q$ is the set of all fuzzy state vectors. Note that the notations for (2) and those for (1) are similar with the only difference being the symbol $\circ$ not adopted in (2). This will be the case for the remaining notations for the EFDES in this section—all the notations for the EFDES will not use this symbol to distinguish them from their FDES counterparts. The state vector $q_k$ at time instance $k$ is mathematically represented as

$$q_k = [kV_1, kV_2, \ldots, kV_N].$$

The variable $q_0$ is used to signify the initial state vector. Here, $N$ is the dimension of the state vector. Unlike the FDES, the state variable $kV_i$ is generally a fuzzy set, but can be a number in [0, 1] or an interval as special cases (in the rest of the paper, an interval or an interval number will mean a special type-1 fuzzy set whose membership function is always 1 over the interval and 0 outside the interval). The universe of discourse of $kV_i$ (i.e., the $x$-axis) is [0, 100%] representing all the possible likelihoods that the EFDES can be in state $i$. The vertical axis of $kV_i$ is [0, 1] to represent the degree of truth. A linguistic label may be assigned...
to each state variable for easier representation and interpretation (e.g., the system is in the state “high” with possibility \(kV_i\)).

The set of all events is \(\Sigma\) and the event \(\sigma_j \in \Sigma\) is formally represented by a fuzzy event transition matrix

\[
\sigma_j = \begin{bmatrix}
\delta_{A_{11}} & \ldots & \delta_{A_{1N}} \\
\vdots & \ddots & \vdots \\
\delta_{A_{N1}} & \ldots & \delta_{A_{NN}}
\end{bmatrix}, \quad 1 \leq j \leq M
\]

where \(M\) is the total number of events. \(\delta_{A_{mn}}, 1 \leq m, n \leq N\), is a fuzzy set characterizing the behavior of the EFDES moving from state \(m\) to state \(n\) when the \(j\)th event takes place. The universe of discourse of \(\delta_{A_{mn}}\) is \([0, 100]\%)\) for the possibility for the system’s transition between the two states. The vertical axis ranges from 0 to 1 to show the degree of truth. Each of state \(m\) and state \(n\) has a linguistic label. In the matrix row direction and the column direction, they should be consistent with the linguistic labels of the corresponding state vectors. Note that in our previous effort [10], all the elements in the event transition matrices are real numbers in \([0, 1]\) (i.e., membership grades), not fuzzy sets. We should also mention that an event itself is always crisp—it either takes place or does not happen. But unlike the conventional DES where the system can be in only one state for any given time, an FDES or EFDES is allowed to be partially in more than one state at the same time.

This is realized by fuzzy event transition matrices.

The parameter \(\delta\) in (2) is a transition mapping that describes how to generate a new state vector from the current fuzzy state vector and a fuzzy event transition matrix. Like the FDES theory, the mapping is mathematically described by

\[
\delta: Q \circ \Sigma \rightarrow Q
\]

and the new fuzzy state vector can be computed by \(q_{k+1} = q_k \circ \sigma_j\). Unlike the FDES theory though, the arguments of the operation \(\circ\) are no longer restricted to numbers. Instead, they can be fuzzy sets.

Through the aforementioned extensions, it should be obvious that the FDES theory is contained in the EFDES theory as a special case because a crisp number is a special case of a fuzzy set.

2) Computation of New Fuzzy States: In \(q_{k+1} = q_k \circ \sigma_j\), the arguments of the operation \(\circ\) can be fuzzy sets or their special cases—interval numbers and numbers (an interval should be expressed as a fuzzy set whose membership is constant 1 over the interval while a number should be expressed as a singleton fuzzy set). Different kinds of fuzzy sets may be used in a mixture fashion. The most common operations are the max–product operation and the max–min operation. To avoid possible confusion regarding the type of the arguments, we use \(\max_q – \text{product}_q\) (i.e., \(\max_q(\cdot)\) and \(\text{product}_q(\cdot)\)) and \(\max_q – \min_q\) when the arguments are fuzzy sets, and \(\max_q – \text{product}_q\) and \(\max_q – \min_q\) when the arguments are interval numbers. When the arguments are numbers, just max–product and max–min will be utilized, which are the conventions in the literature.

When fuzzy set arguments are involved, \(k+1V_n\) in \(q_{k+1}\) can be calculated by

\[
k+1V_n = \max_{1 \leq i \leq N} (kV_i \times \delta_{A_{in}})
\]

\[
= \max_q(kV_1 \times \delta_{A_{1n}}, kV_2 \times \delta_{A_{2n}}, \ldots, kV_N \times \delta_{A_{Nn}})
\]

(3)

where \(\times\) means \(\text{product}_q\). Alternatively, if the \(\max_q – \text{min}_q\) operation is preferred, \(k+1V_n\) is obtained by

\[
k+1V_n = \max_{1 \leq i \leq N} (\min_q(kV_i, \delta_{A_{in}}))
\]

\[
= \max_q(\min_q(kV_1, \delta_{A_{1n}}), \min_q(kV_2, \delta_{A_{2n}}), \ldots, \min_q(kV_N, \delta_{A_{Nn}}))
\]

(4)

To carry out the operations of \(\text{product}_q(\cdot)\), \(\max_q(\cdot)\), and \(\min_q(\cdot)\), the arithmetic based on Zadeh’s extension principle may be employed [28]. Generally speaking, however, using a computer program to automatically produce a closed-form mathematical solution can be very challenging, if not impossible, if the fuzzy sets are allowed to be of arbitrary functions. This is primarily due to the nature of the method in which the membership functions of the fuzzy sets have to be compared with each other to determine which function is the largest or smallest and in which parts of the universes of discourse. Alternatively, one may discretize the universes of discourse of the fuzzy sets so that an approximate numerical solution can be generated by a computer program in an automation fashion.

While permitting \(kV_i\) and \(\delta_{A_{mn}}\) to be arbitrary fuzzy sets may be desirable from the standpoint of a theory and its comprehensiveness, it may not be reasonable for practical applications. Take the theory of fuzzy control and fuzzy modeling as an example. Theoretically speaking, arbitrary fuzzy sets are allowed, but only certain types of fuzzy sets (i.e., fuzzy numbers) are practically sensible, and hence have been heavily used in almost all the studies in the literature. Fuzzy numbers are a class of general fuzzy sets that satisfy the following four requirements [27]: 1) it is a normal (continuous) type-1 fuzzy set (i.e., its support set is bounded); 2) its support set is bounded; 3) its \(\alpha\)-cut set is a closed interval; and 4) the support set of its strong 0-cut set is bounded. The restrictions induced by the requirements are quite mild, but the benefit is substantial—eliminating many of the general fuzzy sets that do not make practical sense in the first place. Obviously, there are an infinite number of types of fuzzy numbers, all of which meet these four requirements. Using fuzzy numbers for \(kV_i\) and \(\delta_{A_{mn}}\) should be adequate not only because they are very diverse and flexible to represent various knowledge and uncertainties, but also because they have clearer and more intuitive meanings than fuzzy sets of arbitrary shapes do. Widely used fuzzy number types include the triangular and the trapezoidal. Any type of fuzzy numbers is allowed in the EFDES theory and a specific choice depends on the system designer.

An even more important advantage of using fuzzy numbers for \(kV_i\) and \(\delta_{A_{mn}}\), as opposed to general fuzzy sets is that it is much easier to obtain closed-form mathematical results for
the operations (3) and (4) using an arithmetic procedure that is based on the \( \alpha \)-cut set approach [28]. Better yet, the procedure can be automatically executed by a computer program using a software package capable of symbolic mathematical operations (e.g., Mathematica and Maple software on the market). We point out that even though the complexity of their execution is significantly different, the extension principle-based arithmetic and the \( \alpha \)-cut-set-based approach generate the same result when fuzzy numbers are involved. We now outline the latter method.

According to the first decomposition theorem [27] in fuzzy set theory, a continuous fuzzy number \( X \) can be represented as the union of its \( \alpha \)-cut sets

\[
X = \bigcup_{\alpha \in [0, 1]} \alpha \otimes [\bar{x}(\alpha), \bar{\pi}(\alpha)]
\]

where \([\bar{x}(\alpha), \bar{\pi}(\alpha)]\) is an \( \alpha \)-cut set of \( X \), which is an interval (i.e., an interval number) containing all the elements in the universe of discourse of \( X \) whose membership grades are greater than or equal to the specified membership grade \( \alpha \). Here, \( \bar{x}(\alpha) \) and \( \bar{\pi}(\alpha) \) are the lower and upper terminal points whose values depend on \( \alpha \). And \( \bigcup \) is the Zadeh fuzzy union operation \([i.e., \max()]\) and \( \alpha \otimes [\bar{x}(\alpha), \bar{\pi}(\alpha)] \) means that the membership grade for \( [x(\alpha), \pi(\alpha)] \) is \( \alpha \).

Based on the \( \alpha \)-cut representation, the operations of fuzzy numbers \( \times \) [i.e., products()] \( \max() \), and \( \min() \) in (3) and (4) can be expressed as [27]

\[
k^V_i \times \lambda_{1n} = \bigcup_{\alpha \in [0, 1]} \alpha \otimes ([k_{\bar{x}}(\alpha), k_{\bar{\pi}}(\alpha)] \times [\lambda_{1n}(\alpha), \lambda_{1n}(\alpha)])
\]

\[
\max_S(k^V_i, \lambda_{1n}) = \bigcup_{\alpha \in [0, 1]} \alpha \otimes \max([k_{\bar{x}}(\alpha), k_{\bar{\pi}}(\alpha)], [\lambda_{1n}(\alpha), \lambda_{1n}(\alpha)])
\]

\[
\min_S(k^V_i, \lambda_{1n}) = \bigcup_{\alpha \in [0, 1]} \alpha \otimes \min([k_{\bar{x}}(\alpha), k_{\bar{\pi}}(\alpha)], [\lambda_{1n}(\alpha), \lambda_{1n}(\alpha)])
\]

where \( k_{\bar{x}}(\alpha), k_{\bar{\pi}}(\alpha), \lambda_{1n}(\alpha), \) and \( \lambda_{1n}(\alpha) \) are numbers in \([0, 1]\), and \( \times \) in (5) is the multiplication operation for interval numbers. According to the interval number operations [27], the previous expressions lead to the following:

\[
k^V_i \times \lambda_{1n} = \bigcup_{\alpha \in [0, 1]} \alpha \otimes [k_{\bar{x}}(\alpha) \times \lambda_{1n}(\alpha), k_{\bar{\pi}}(\alpha) \times \lambda_{1n}(\alpha)]
\]

\[
\max_S(k^V_i, \lambda_{1n}) = \bigcup_{\alpha \in [0, 1]} \alpha \otimes \max(k_{\bar{x}}(\alpha), \lambda_{1n}(\alpha), k_{\bar{\pi}}(\alpha), \lambda_{1n}(\alpha))
\]

\[
\min_S(k^V_i, \lambda_{1n}) = \bigcup_{\alpha \in [0, 1]} \alpha \otimes \min(k_{\bar{x}}(\alpha), \lambda_{1n}(\alpha), k_{\bar{\pi}}(\alpha), \lambda_{1n}(\alpha))
\]

We point out that because fuzzy numbers can be represented as the union of their \( \alpha \)-cut sets and this representation can be realized by a symbolic mathematical software package automatically and quickly with the result stored in the computer prior to the computations in (5)–(12), the computation burden of this approach is low.

3) Construction of Fuzzy Event Transition Matrices: Fuzzy event transition matrices play a central role in the theory of EFDES. They are the core of EFDES and capture and represent much of expert knowledge/experience that may be subjective, vague, and imprecise in practical applications. Furthermore, they are used to compute future states from the present one for a particular system variable of interest, say \( x \) (e.g., potency). The process for generating a fuzzy event transition matrix in EFDES is more complex than that in FDES partially because fuzzy sets, \( \lambda_{mn} \), are now needed to be produced to represent the fuzzy probabilities in the end of the process instead of crisp numbers in the case of FDES. To construct a fuzzy event transition matrix is to determine all the fuzzy sets, each of which is an element of the matrix \( \lambda_{mn} \). The elements should capture expert knowledge on state transitions and represent the possibility of the system moving from one state to another when a system event happens (e.g., patient is treated by a regimen).

For an \( N \times N \) event transition matrix, there are \( N \) different states. The element \( \lambda_{mn} \) indicates the fuzzy possibility for the system variable to transfer from state \( m \) (supposedly “medium”) to state \( n \) (supposedly “high”). The system developer may use real (measurement) data to compute the fuzzy possibility. However, this can be difficult because the states themselves are only vaguely defined (e.g., “medium” and “high”); thus, quantifying the possibility can be challenging. Additionally, the real data may not be available in many real-world applications. Alternatively, the developer may determine the fuzzy possibilities in a more subjective manner. One possible approach is to directly write out the fuzzy sets (or in special cases, membership grades and interval numbers) in an ad hoc fashion. Nevertheless, it can
be hard to manually generate $N \times N$ fuzzy sets systematically and consistently without contradictions if $N$ is large and/or more than one expert’s knowledge is to be captured but experts with conflicting or opposite opinions disagree with one another without consensus (a norm in medicine). To resolve the difficulty, we propose the following systematic method for determining $\Lambda_{m,n}$. With the help of a group of domain experts, the system developer first defines a fuzzy set, designated as $D$, over the universe of discourse of $x$, denoted by $X$, that represents the experts’ knowledge concerning the possibility of the system transitioning from state $m$ to state $n$ for different values of $x$ (say, potency) after the $j$th event occurs. (For instance, $D$ can be a fuzzy set describing the possibility of general patient population moving from “medium” potency state to “high” potency state after they start regimen ABC (i.e., the $j$th event) whose potency ranges from, say, 70% to 95%). This fuzzy set can be either type-1 or type-2, depending on the extent of experts’ disagreement. It does not need to be defined over the entire universe of discourse; only part of the universe of interest would suffice (e.g., 60%–100% in the aforementioned example). Then, the developer needs to assign a value, designated as $F$, which can be a crisp number, an interval, a type-1 fuzzy set, or a type-2 fuzzy set, to $x$ (i.e., the potency value of regimen ABC) that is associated with the particular event.

Without loss of generality, assume that $D$ is a type-2 fuzzy set represented by a 3-D membership function $z = \varphi_D(x, y)$, where $x \in X$ and $y$ is a variable representing the membership grade for $x$. The relationship between $x$ and $y$ is characterized by two membership functions (instead of one in the case of type-1 fuzzy sets)—an upper membership function and a lower membership function. For a given value of $x = x'$, the upper and lower membership functions will intersect with the vertical line of $x = x'$, and respectively produce an upper point and a lower terminal point of an interval of membership grades. The interval is denoted by $\mu^D_{x'}$. Obviously, $\mu^D_{x'}$ is a subinterval of $[0, 1]$ (i.e., $\mu^D_{x'} \subseteq [0, 1]$). A type-1 fuzzy set can be defined over $\mu^D_{x'}$, which mathematically is $z = \varphi_D(x', y)$. Therefore, $z$ represents the membership of $y$. The collection of all these type-1 fuzzy sets for the entire universe of discourse of $x$ is called a secondary membership function [29], which is $z = \varphi_D(x', y)$. All the intervals $\mu^D_{x'}$ due to different values of $x$ form so-called primary memberships $\mu^D_x$ (obviously, $\mu^D_x \subseteq [0, 1]$).

The collection of all the left and right terminal points of $\mu^D_{x'}$ forms the upper and lower membership functions, respectively. Note that the two functions along with the area bounded by them are the primary memberships. Formally, $D$ can be represented by

$$D = \{ \varphi_D(x, y) | \forall x \in X, \forall y \in \mu^D_{x'} \}. \quad (13)$$

In the interest of generality, we suppose that $F$ is a type-2 fuzzy set (crisp number, interval number, and type-1 fuzzy set are all its special cases)

$$F = \{ \varphi_F(x, y) | \forall x \in X, \forall y \in \mu^F_{x} \}. \quad (14)$$

We then apply a fuzzy intersection operator on $D$ and $F$. Generally speaking, the result is a type-2 fuzzy set. Denote it by $T$ that is calculated according to [30]

$$T = \{ \varphi_T(x, y) | \varphi_T(x, y) = \sup_{y = y^D \cap y^F} (\varphi_D(x, y^D) \cap \varphi_F(x, y^F)), \forall x \in X \}$$

$$= \bigcup_{\mu^D_x \subseteq [0, 1]} \left( \bigcap_{\mu^F_y \subseteq [0, 1]} \left( \sup_{y = y^D \cap y^F} \left( \varphi_D(x, y^D) \cap \varphi_F(x, y^F) \right) \right) \right)$$

$$= \bigcup_{\mu^D_x \subseteq [0, 1]} \left( \bigcap_{\mu^F_y \subseteq [0, 1]} \left( \sup_{y = y^D \cap y^F} \left( \varphi_D(x, y^D) \cap \varphi_F(x, y^F) \right) \right) \right)$$

$$= \mu^F_{x} \subseteq [0, 1] \quad (15)$$

where $\cap$ can be any fuzzy AND operator such as the Zadeh AND operator [29] and $\sup()$ is the operation to pick up the largest element (i.e., the $\max()$ operator). The primary membership grade $y (= y^D \cap y^F)$ and the secondary membership grade $= \varphi_D(x, y^D) \cap \varphi_F(x, y^F)$ are computed. If more than one value of $\varphi_D(x, y^D) \cap \varphi_F(x, y^F)$ is produced for the same value of $y$, the largest value will be selected as the result by the $\sup()$ operation.

A fuzzy set type reducer is then needed to convert $T$ to a type-1 fuzzy set, which is $\Lambda_{m,n}$. Note that $\Lambda_{m,n}$ represents the relationship between $y$ and $z$, instead of $x$ and $y$ needed for other type-2 applications in the literature. There exist several type reducers, and we modify the so-called height type-reduction method for our purpose. Consequently, the membership function for $\Lambda_{m,n}$, denoted by $\Lambda_{m,n}(y)$, defined over the universe of discourse $[0, 1]$ is [29]

$$\Lambda_{m,n}(y) = \sup_x \varphi_T(x, y). \quad (16)$$

Since $F$ can be a crisp number, an interval, a type-1 fuzzy set, or a type-2 fuzzy set, and $D$ can be a type-1 fuzzy set or a type-2 fuzzy set, there exist eight combinations. Formula (15) covers the most general case—both $D$ and $F$ are type-2 fuzzy sets. The formulas for the rest of the seven situations can be derived from (15) as a special case. The results are summarized in Table I. Note that situation No. 8 is the FDES.

There lacks a method in the current literature to produce a solution for (15) when both $D$ and $F$ are general type-2 fuzzy sets. If they are interval type-2 fuzzy sets, numerical solutions can be obtained via algorithms for situations Nos. 1–5 [29]. The remaining three situations are readily computable as they involve type-1 fuzzy sets only. For better illustration, in Section III, we will use the HIV/AIDS example to demonstrate how to carry out (15) and (16) for situations Nos. 4 and 6.

4) Parallel Composition of EFDESs: Like the DES and FDES theories, to model a (complex) EFDES, a convenient way is often to model each subsystem first, and then synthesize them by parallel composition. Since the memberships of all the states and events have been extended from numbers to fuzzy sets, we need to modify the operations of the parallel composition accordingly. Suppose two EFDES are given

$$G_{m_1} = (Q_{m_1}, \Sigma_{m_1}, \delta_{m_1}, q_0) \quad G_{m_2} = (Q_{m_2}, \Sigma_{m_2}, \delta_{m_2}, q_0).$$

Then, through parallel composition $\parallel$, the combined EFDES

$$G_m = (Q_m, \Sigma_m, \delta_m, q_0)$$

is

$$G_m = G_{m_1} \parallel G_{m_2} = (Q_{m_1} \times Q_{m_2}, \Sigma_{m_1} \cup \Sigma_{m_2}, \delta_{m_1} \times \delta_{m_2}, (q_0, q_0)).$$

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where \( m, m_1, \) and \( m_2 \) are positive integers. We now describe how the parallel composition of EFDES is executed.

There are two steps: 1) compute the state vectors for \( G_m \) and 2) compute the fuzzy event transition matrices for \( G_m \). For the first step, assume that \( q_0 \) and \( q'_0 \) are fuzzy state vectors in \( Q_m \) and \( Q_{m_2} \), respectively, and \( h \, V_j \) (\( 1 \leq j \leq N \)) and \( j' \, V_k' \) (\( 1 \leq k \leq N' \)) are elements of \( q_0 \) and \( q'_0 \), respectively. Here, \( h \) and \( i \) are nonnegative integers, and \( j, k, N, \) and \( N' \) are positive integers, where \( N \) and \( N' \) are the dimensions of fuzzy state vectors in \( G_m \) and \( G_{m_2} \), respectively. The dimension of the state vectors in \( G_m \) is one row and \( N \times N' \) column. The composition of the \( j \)-th element in \( q_i \) (i.e., \( h \, V_j \)) with the \( k \)-th element in \( q_i' \) (i.e., \( j' \, V_k' \)) yields the element in the composite state vector located in the column corresponding to \((j, k)\).

In the second step, we determine the event transition matrices of \( G_m \). Designate \( M^*, M, \) and \( M' \) as the total number of events for \( G_m, G_{m_1}, \) and \( G_{m_2} \), respectively. Suppose that \( \beta_i \) and \( \beta_i' \) is an element of the event matrix \( \sigma_{1i} \) (\( 1 \leq \beta_i \leq M \)) in \( G_m \), and \( \beta_i' \) is an element of the event matrix \( \sigma'_{1i} \) (\( 1 \leq \beta_i' \leq M' \)) in \( G_{m_2} \). After the parallel composition, the dimension of \( \sigma_{1i} \) and \( \sigma'_{1i} \) will no longer be the same as that in \( G_m \) and \( G_{m_2} \), respectively. The dimension of all the events in \( G_m \) should be \((N \times N') \times (N \times N') \). Let \( \sigma_{1i}^* \) signify the representation of \( \sigma_{1i} \) in \( G_m \) (discussion concerning \( \sigma_{1i}^* \) would be similar and is omitted for brevity). Suppose that \( \beta_i \) \( \sigma_{1i}^* \) an element in \( \sigma_{1i} \) that represents a concurrent transition from state \( k_1 \) to state \( k_2 \) (\( 1 \leq k_1, k_2 \leq N \)) in \( G_m \) and from state \( j_1 \) to state \( j_2 \) (\( 1 \leq j_1, j_2 \leq N' \)) in \( G_{m_2} \). It can be determined as follows.

The first scenario is that \( G_{m_2} \) does not have the event \( \sigma_{1i} \). Then

\[
\beta_i \, A_{kj}^* = \begin{cases} 
\beta_i \, A_{kj}, & \text{if } j_1 = j_2 \\
Z, & \text{if } j_1 \neq j_2.
\end{cases}
\]

For those \( \beta_i \) \( A_{kj}^* \) corresponding to the unchanged state in \( G_{m_2} \) (i.e., \( j_1 = j_2 \)), the behavior of the event transition from \( k \) to \( j \) in \( G_m \) should be the same as that from \( k_1 \) to \( k_2 \) in \( G_m \). Hence, \( \beta_i \, A_{kj}^* = \beta_i \, A_{kj} \). For those \( \beta_i \) \( A_{kj}^* \) representing state change in \( G_{m_2} \) (i.e., \( j_1 \neq j_2 \)), \( \beta_i \, A_{kj}^* = Z \), where \( Z \) represents a type-1 singleton fuzzy number 0 whose membership is 1 at 0 in the universe of discourse and is 0 elsewhere (by fuzzy number notation, this fuzzy number can be expressed as 1/0). This is because \( G_{m_2} \) does not have the event \( \sigma_{1i} \) hence, any state change in \( G_{m_2} \) should not appear in \( G_m \), and thus \( Z \).

The second scenario is that \( G_{m_2} \) also has the event \( \sigma_{1i} \). Then

\[
\beta_i \, A_{kj}^* = \min_{S} \{ \beta_i \, A_{kj}, \beta_i \, A_{kj}^* \}.
\]

This is because \( \sigma_{1i} \) now takes place in both \( G_m \) and \( G_{m_2} \) to the extents of \( \beta_i \, A_{kj} \) and \( \beta_i \, A_{kj}^* \), respectively. Therefore, the fuzzy and operator should be applied (e.g., the Zadeh and operator used here).

We point out that state explosion problem due to parallel composition is universal to all DESS and EFDES is no exception.
Many methods have been proposed in the DES literature to address this problem. They include modular control, decentralized control (e.g., [31]), hierarchical control [32], online control with limited and variable look-ahead policies [33], and various techniques for model abstractions. These methods can also be used in EFDES.

III. ILLUSTRATIVE EXAMPLES

To make the aforementioned theoretical development easier to understand, we now provide three detailed numerical examples that are related to medical applications.

Example 1: Construct fuzzy event transition matrices of EFDES for HIV/AIDS treatment. Continue the HIV/AIDS treatment application mentioned in Section I. We will consider two of the eight scenarios in Table I (more specifically, situations Nos. 4 and 6).

There were three states for regimen’s potency—Initial, Medium, and High. When the FDES theory was applied to a treatment-naïve patient, the state vector was

\[
\mathbf{q}_0 = \begin{bmatrix} \text{Initial} & \text{Medium} & \text{High} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]

which means that the patient was not in Medium or High potency state but in the Initial state because he/she had never received treatment before. For clearer illustration, we list the state names on the top of the vector. Using the EFDES theory presented before, the three crisp membership values become three fuzzy numbers as follows:

\[
\mathbf{q}_0 = \begin{bmatrix} \text{Initial} & \text{Medium} & \text{High} \\
\text{Pattern } 0 & \text{Pattern } 1 & \text{Pattern } 2 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

where \(0 = 1/1\) and \(0 = 0 = 0 = Z\). \(Z\) is the singleton fuzzy number defined in Section II-B4 while \(1/1\) represents a type-1 singleton fuzzy number 1. One may use other fuzzy numbers 0 and 1 (e.g., the trapezoidal function), which will likely make the example complicated in computation.

Now, we need to create the fuzzy event transition matrix for potency, which is \(3 \times 3\). The process for the FDES is illustrated in Fig. 1 where the regimen’s potency is assumed to be 85%. Using the two membership grades in Fig. 1 (i.e., the intersections of the vertical line of 85% potency and the two fuzzy sets) can construct the fuzzy event transition matrix for the FDES as

\[
\mathbf{\sigma}_1 = \begin{bmatrix} \text{Initial} & \text{Medium} & \text{High} \\
0 & 0.4296 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

The corresponding fuzzy event transition matrix \(\mathbf{\sigma}_1\) for the EFDES is

\[
\mathbf{\sigma}_1 = \begin{bmatrix} \text{Initial} & \text{Medium} & \text{High} \\
\text{Pattern } 0 & \text{Pattern } 1 & \text{Pattern } 2 \\
Z & \{} & \{} \\
Z & \{} & \{} \\
Z & \{} & \{} \\
\end{bmatrix}
\]

\(\mathbf{\sigma}_1\) 32

We now discuss how to produce the fuzzy numbers \(\text{Pattern } 0\) and \(\text{Pattern } 1\) under different conditions.

A. First Scenario

The first scenario is that a group of experts all agree to use the same type-1 fuzzy sets [corresponding to the fuzzy set \(\text{Pattern } 0\) in (13)] “Medium” and “High” for potency (the fuzzy sets describe the possibilities of the system changing from the initial state to “Medium” or “High” state). But they fail to reach a consensus on the exact potency value of the regimen. Hence, the regimen’s potency should not be a crisp number but a fuzzy number. This is situation No. 6 in Table I. Fig. 2 shows such a hypothetical example where a symmetrical triangular fuzzy number \(\text{Pattern } 0\) centered on 85% is used to represent diverse opinions of the experts on the potency value [corresponding to the fuzzy set \(\text{T} \) in (14)]. The membership function of \(\text{Pattern } 0\) is

\[
\mu_{\text{Pattern } 0}(x) = \begin{cases} 
0.82 - x, & 0.82 < x < 0.85 \\
0.03, & 0.85 < x < 0.8703 \\
0, & \text{elsewhere in } [0, 1] 
\end{cases}
\]

where \(x\) is the potency value. Using Table I, one can obtain \(\text{Pattern } 0\). By applying an intersection operation (\(\text{Pattern } 0\)) is used here), two fuzzy numbers, \(\text{Pattern } 1\) and \(\text{Pattern } 1\), are resulted [corresponding to the fuzzy set \(\text{T} \) computed by (15)]. \(\text{Pattern } 1\) is produced by intersecting \(\text{Pattern } 0\) with “Medium” point by point along the potency axis. The membership function of the resulting \(\text{Pattern } 1\), shown as a bold curve in Fig. 2, is

\[
\mu_{\text{Pattern } 1}(x) = \begin{cases} 
0.82 - x, & 0.82 < x < 0.85 \\
0.854 < x < 0.8703 \\
0.8703 < x < \text{elsewhere in } [0, 1].
\end{cases}
\]
The minimal and maximal membership values of $\mu_{P_M}(x)$ are 0 and 0.5138, respectively (Fig. 2). Therefore, according to Table I, the fuzzy number $A_{12}$ is an interval number $[0, 0.5138]$. In our case, $P_{H}$ happens to be identical to $P$ (thus, it is not marked in Fig. 2). Subsequently, the minimal and maximal membership values after the intersection are 0 and 1, respectively, which means that $A_{13}$ is an interval number $[0, 1]$. Hence, the fuzzy event transition matrix $\sigma_1$ in (21) is

$$
\sigma_1 = \begin{bmatrix}
Z & [0, 0.5138] & [0, 1] \\
Z & Z & Z \\
Z & Z & Z 
\end{bmatrix}.
$$

Compared to the FDES event transition matrix $\tilde{\sigma}_1$ in (20), the nonzero elements in (22) are interval numbers instead of crisp numbers. The practical implication is that the uncertainty on the exact potency value has been conveniently expressed by the domain expert (i.e., the physician), and has been effectively processed by the EFDES approach.

### B. Second Scenario

This scenario is the opposite of the first scenario—the experts agree to use the exact same (crisp) value to represent the regimen’s potency, but they have different definitions for the term “Medium” (and/or “High”). If the inclusion of all the different opinions of the experts is desired, “Medium” and “High” should be defined as type-2 fuzzy sets [29]. This is situation no. 4 in Table I. Fig. 3 shows an example. The primary memberships of a type-2 fuzzy set consists of an upper membership function and a lower membership function to bound a range of opinions technically called footprint of uncertainty [Fig. 3(a)]. The primary memberships for “Medium” and “High” are mathematically defined in Table I, and the two bounded regions (i.e., the footprints of uncertainty) are shown as two gray areas with distinct intensities in Fig. 3(a).

According to Table I, we can obtain $A_{12}$ and $A_{13}$ as follows. Draw a vertical line at a potency value [85% in Fig. 3(a)]. The line will intersect with the upper and lower membership functions of the two type-2 fuzzy sets, resulting in two intervals of the membership grades—one for “Medium” and the other for “High.” They are actually two results after applying the Zadeh fuzzy intersection operation to “Medium” and the 85% potency line, and “High” and the same line, respectively. The intervals then serve as the universes of discourse for secondary membership functions of the respective type-2 fuzzy sets at the potency value where the vertical line is put. A secondary membership function describes the likelihood whether each of the membership grades will take place. Like the primary memberships and footprint of uncertainty, the secondary membership functions are developed by the experts/developer. For instance, they may be defined in such a way to reflect the experts’ individual preference/bias exhibited when they define the primary memberships. In Fig. 3(b), we suppose that the secondary membership function for “Medium” at 85% potency is of the triangular type

$$
\varphi_M(y) = \begin{cases} 
\frac{2(y - y_M)}{y_M - y_M}, & y_M < y < \frac{y_M + y_M}{2} \\
\frac{2(y_M - y)}{y_M - y_M}, & \frac{y_M + y_M}{2} \leq y < y_M \\
0, & \text{elsewhere in } [0, 1]
\end{cases}
$$

for “Medium” at 85% potency is of the triangular type.
be triangular
\[
\varphi_{H}(y) = \begin{cases} 
\frac{2(y - y_H)}{\sigma_H^2} & \text{if } y < y_H < \frac{y_H + y_{TT}}{2} \\
\frac{2(y_{TT} - y)}{\sigma_H^2} & \text{if } y < \frac{y_H + y_{TT}}{2} < y < y_{TT} \\
0 & \text{elsewhere in } [0, 1].
\end{cases}
\]

Note that \(y_M, \sigma_M, y_H, \text{ and } y_{TT}\) are marked in Fig. 3(a) and their values can be determined using the functions in Table II. Given potency of 85%, \(y_M = 0.3247, \sigma_M = 0.5461, y_H = 0.9802, \text{ and } y_{TT} = 1\).

Assigning \(\varphi_M(y)\) and \(\varphi_H(y)\) to the fuzzy numbers \(1^1A_{12}\) and \(1^1A_{13}\), respectively, completes the process of producing the matrix. The fuzzy number \(1^1A_{12}\) is triangular
\[
\varphi_{1A_{12}}(y) = \begin{cases} 
9.0334y - 2.9331, & 0.3247 < y < 0.4354 \\
4.9331 - 9.0334y, & 0.4354 < y < 0.5461 \\
0, & \text{elsewhere in } [0, 1]
\end{cases}
\]

and the fuzzy number \(1^1A_{13}\) is also triangular
\[
\varphi_{1A_{13}}(y) = \begin{cases} 
101.0101y - 99.0101, & 0.9802 < y < 0.9901 \\
101.0101 - 101.0101y, & 0.9901 < y < 1 \\
0, & \text{elsewhere in } [0, 1]
\end{cases}
\]

Consequently, the fuzzy event transition matrix \(\sigma_1\) is
\[
\sigma_1 = \begin{bmatrix} 
Z & \varphi_{1A_{12}} & \varphi_{1A_{13}} \\
Z & Z & Z \\
Z & Z & Z 
\end{bmatrix}.
\] (23)

In contrast to the FDES event transition matrix \(\tilde{\sigma}_1\) in (20) where all the elements are crisp numbers, all the elements in (23) are fuzzy numbers. From practice perspective, the advantage is substantial—the EFDES method is capable of simultaneously capturing and representing a range of opinions contributed by a group of domain experts that do not agree with one another. This is something unachievable under the framework of FDES.

So far, we have considered the case when the type-2 fuzzy sets are general in the sense that their primary memberships and secondary membership functions can be arbitrary but reasonable shapes. If the fuzzy set is an interval type-2 fuzzy set, which can be in the case if the experts want to treat the likelihoods of membership grades for a parameter of interest (e.g., potency) at each parameter value equally or uniformly, we can simplify the aforementioned results by letting \(\varphi_M(y) = 1\) or \(\varphi_H(y) = 1\). To make a concrete example, if the secondary membership function for “Medium” is as shown in Fig. 3(b), then the fuzzy number \(1^2A_{12}\) reduces to an interval
\[
\varphi_{1A_{12}}(y) = [0.3247, 0.5461].
\]

Example 2: Computing new fuzzy states using fuzzy event transition matrices. Assume that the fuzzy event transition matrix is (23). It can then be used to compute new fuzzy state vector \(q_1\) from the current state, which is the initial state \(q_0\) in (19)
\[
q_1 = q_0 \circ \sigma_1 = \begin{bmatrix} \frac{1}{1} & Z & Z \end{bmatrix} \circ \begin{bmatrix} Z & 1^1A_{12} & 1^1A_{13} \\
Z & Z & Z \\
Z & Z & Z 
\end{bmatrix} = \begin{bmatrix} Z & 1^1A_{12} & 1^1A_{13} \end{bmatrix}.
\]

The \(\text{max}_S\)-product\(_S\) operation is used for \(\circ\) as all the elements are type-1 fuzzy numbers.

To make this example nontrivial, let us now assume that the HIV/AIDS patient has failed the first-round treatment and must receive a second-round treatment. Without loss of generality, suppose the potency event transition matrix for a second-round regimen that the patient takes is obtained via one of the two approaches described earlier
\[
\sigma_2 = \begin{bmatrix} Z & Z & Z \\
Z & 2^1A_{22} & Z \\
Z & 2^1A_{32} & 2^1A_{33} 
\end{bmatrix}
\]

where the type-1 fuzzy numbers \(2^1A_{22}, 2^1A_{32}, \text{ and } 2^1A_{33}\) have the following triangular membership functions:
\[
\varphi_{2A_{12}}(y) = \begin{cases} 
10y - 2, & 0.2 < y < 0.3 \\
4 - 10y, & 0.3 < y < 0.4 \\
0, & \text{elsewhere in } [0, 1]
\end{cases}
\]

\[
\varphi_{2A_{13}}(y) = \begin{cases} 
10y - 6, & 0.6 < y < 0.7 \\
8 - 10y, & 0.7 < y < 0.8 \\
0, & \text{elsewhere in } [0, 1]
\end{cases}
\]

\[
\varphi_{2A_{22}}(y) = \begin{cases} 
10y - 8, & 0.8 < y < 0.9 \\
10 - 10y, & 0.9 < y < 1 \\
0, & \text{elsewhere in } [0, 1]
\end{cases}
\]

The new potency state after the patient is treated is
\[
q_2 = q_1 \circ \sigma_2 = \begin{bmatrix} Z & 1^1A_{12} & 1^1A_{13} \end{bmatrix} \circ \begin{bmatrix} Z & Z & Z \\
Z & 2^1A_{22} & Z \\
Z & 2^1A_{32} & 2^1A_{33} 
\end{bmatrix} = \begin{bmatrix} Z & 1^1A_{12} \times 2^1A_{22}, 1^1A_{13} \times 2^1A_{32} \end{bmatrix} \times 1^1A_{13} \times 2^1A_{33} = \begin{bmatrix} Z & 2^1A_{22} & 2^1A_{32} \\
Z & 2^1A_{32} & 2^1A_{33} 
\end{bmatrix}.
\]
Fig. 4. Graphical representations of the triangular fuzzy numbers $^2\nu_2$ and $^2\nu_3$.

where $^2\nu_2$ and $^2\nu_3$ are type-1 fuzzy numbers whose membership functions are (Fig. 4)

\[
\varphi_{^2\nu_2}(y) = \begin{cases}
\sqrt{2162.7 + 1010.1y} - 52.5051, & 0.5881 < y < 0.6931 \\
-\sqrt{2162.7 + 1010.1y} + 54.5051, & 0.6931 \leq y < 0.8 \\
0, & \text{elsewhere in } [0, 1]
\end{cases}
\]

\[
\varphi_{^2\nu_3}(y) = \begin{cases}
\sqrt{2070.7 + 1010.1y} - 53.5051, & 0.7842 < y < 0.8911 \\
-\sqrt{2070.7 + 1010.1y} + 55.5051, & 0.8911 \leq y < 1 \\
0, & \text{elsewhere in } [0, 1].
\end{cases}
\]

Using the key building blocks illustrated in Examples 1 and 2, we are developing an EFDES for HIV/AIDS treatment that will include a system validation against the actual patient treatment records. A complete description of the system is beyond the scope and space limit of this paper. It would require a full-length paper as was the case for our FDES HIV/AIDS system [24].

Example 3: Parallel composition of EFDESs. This example is unrelated to the first two examples and HIV/AIDS. However, in a sense it is realistic in that it is related to two important human organs—heart and lung. Suppose an EFDES is composed of two subsystems $G_1$ and $G_2$. $G_1$ describes patient’s lung condition and $G_2$ heart condition. In $G_1$, the lung condition is classified as “Excellent” (E, for short), “Fair” (F), and “Poor” (P), which is represented by a fuzzy state vector

\[
q_k = [kV_1, kV_2, kV_3]
\]

where $kV_1$, $kV_2$, and $kV_3$ are fuzzy numbers charactering the states “Excellent,” “Fair,” and “Poor,” respectively. There are six events denoted as $\sigma_j$, $j = 1, \ldots, 6$. Patient’s heart condition is also classified as “Excellent” ($E'$, for short), “Fair” ($F'$), and “Poor” ($P'$), which is described by a fuzzy state vector in $G_2$

\[
q_k' = [IV_1', IV_2', IV_3']
\]

The three elements are fuzzy numbers concerning “Excellent,” “Fair,” and “Poor.” Suppose that there are six events in total for the heart as well; they are labeled as $a_n$, $n = 1, \ldots, 6$. Also, suppose that $G_1$ and $G_2$ do not share a common event, i.e., the treatment will not affect heart and lung simultaneously.

The dimension of the resultant state vector of $G_1 || G_2$ is 9 (i.e., $3 \times 3$) that covers the following nine combinations of the states: ($E$, $E'$), ($E$, $F'$), ($E$, $P'$), ($F$, $E'$), ($F$, $F'$), ($F$, $P'$), ($P$, $E'$), ($P$, $F'$), and ($P$, $P'$). The value of each composite state can be calculated according to (17)

\[
(q_k, q_k') = [(E, E'), (E, F'), (E, P'), (F, E'), (F, F'), (F, P'), (P, E'), (P, F'), (P, P')]
\]

\[
= \left[\min_S(kV_1, IV_1'), \min_S(kV_1, IV_2'), \min_S(kV_1, IV_3'), \min_S(kV_2, IV_1'), \min_S(kV_2, IV_2'), \min_S(kV_2, IV_3'), \min_S(kV_3, IV_1'), \min_S(kV_3, IV_2'), \min_S(kV_3, IV_3')\right].
\]

Next, we reformulate all the event transition matrices in $G_1$ and $G_2$. Without loss of generality, let us focus on event $\sigma_1$ only, which is related to change of lung condition from “Excellent” to “Fair.” It is represented by the following fuzzy transition matrix:

\[
\sigma_1 = \begin{bmatrix}
1B_{11} & 1B_{12} & 1B_{13} \\
1B_{21} & 1B_{22} & 1B_{23} \\
1B_{31} & 1B_{32} & 1B_{33}
\end{bmatrix}
\]

Elements in $\sigma_1^*$, which is $\sigma_1$ after $G_1 || G_2$, can be computed using (18) since $G_2$ does not contain event $\sigma_1$. The result is, (15), as shown at the bottom of this page.
For better illustration, we list all the state combinations on the top and right side of the matrix. We now explain how the elements are obtained via (18). Take the situation when lung condition changes from “Excellent” to “Fair” but heart condition remains the same as an example. It involves three elements in $\sigma_1^i : (E, E^i) \rightarrow (F, E^i)$, $(E, F^i) \rightarrow (F, F^i)$, and $(E, P^i) \rightarrow (F, P^i)$. Clearly, the value of all these three elements is $1^{B_{21}}$ according to (18). On the other hand, consider the situation when both lung and heart conditions change from “Excellent” to “Fair.” It involves only one element in $\sigma_1^i$ corresponding to $(E, E^i) \rightarrow (F, F^i)$. Given that $j_1 \neq j_2$ and $G_2$ does not contain event $\sigma_1$, extent for the behavior of event transition $(E, E^i) \rightarrow (F, F^i)$ in $G_m$ should be $Z$.

IV. CONCLUSION

We have extended the theory of FDES to a theory of EFDES that can effectively deal with the two general and critically important issues concerning the representation and acquisition of experts’ knowledge and subjectivity. Under the new framework, elements in the fuzzy state vectors and fuzzy event transition matrices can be type-1 fuzzy numbers as opposed to numbers in the original FDES theory. The extended theory not only enables domain experts to more conveniently and realistically express their uncertain and imprecise knowledge and subjective judgments but also makes their capture and representation possible. We have established mathematical operations that cover the construction of the event transition matrices and the computation of the new fuzzy states and new fuzzy event transitions. Furthermore, we have developed the corresponding parallel composition. Three detailed numerical examples are provided to show how to use the key components of the extended theory.

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His current research interests include the development of novel biomedical acoustic sensor systems, type-2 fuzzy systems, and fuzzy discrete event systems.

For better illustration, we list all the state combinations on the top and right side of the matrix. We now explain how the elements are obtained via (18). Take the situation when lung condition changes from “Excellent” to “Fair” but heart condition remains the same as an example. It involves three elements in $\sigma_1^i : (E, E^i) \rightarrow (F, E^i)$, $(E, F^i) \rightarrow (F, F^i)$, and $(E, P^i) \rightarrow (F, P^i)$. Clearly, the value of all these three elements is $1^{B_{21}}$ according to (18). On the other hand, consider the situation when both lung and heart conditions change from “Excellent” to “Fair.” It involves only one element in $\sigma_1^i$ corresponding to $(E, E^i) \rightarrow (F, F^i)$. Given that $j_1 \neq j_2$ and $G_2$ does not contain event $\sigma_1$, extent for the behavior of event transition $(E, E^i) \rightarrow (F, F^i)$ in $G_m$ should be $Z$.

IV. CONCLUSION

We have extended the theory of FDES to a theory of EFDES that can effectively deal with the two general and critically important issues concerning the representation and acquisition of experts’ knowledge and subjectivity. Under the new framework, elements in the fuzzy state vectors and fuzzy event transition matrices can be type-1 fuzzy numbers as opposed to numbers in the original FDES theory. The extended theory not only enables domain experts to more conveniently and realistically express their uncertain and imprecise knowledge and subjective judgments but also makes their capture and representation possible. We have established mathematical operations that cover the construction of the event transition matrices and the computation of the new fuzzy states and new fuzzy event transitions. Furthermore, we have developed the corresponding parallel composition. Three detailed numerical examples are provided to show how to use the key components of the extended theory.

REFERENCES


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