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A general technique for deriving analytical structure of fuzzy controllers using arbitrary trapezoidal input fuzzy sets and Zadeh AND operator☆

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Abstract

Deriving the analytical structure of fuzzy controllers is very important as it creates a solid foundation for better understanding, insightful analysis, and more effective design of fuzzy control systems. We previously developed a technique for deriving the analytical structure of the fuzzy controllers that use Zadeh fuzzy AND operator and the symmetric, identical trapezoidal or triangular input fuzzy sets. Many fuzzy controllers use arbitrary trapezoidal/triangular input fuzzy sets that are asymmetric. At present, there exists no technique capable of deriving the analytical structure of these fuzzy controllers. Extending our original technique, we now present a novel method that can accomplish rigorously the structure derivation for any fuzzy controller, Mamdani type or TS type, that employs the arbitrary trapezoidal input fuzzy sets and Zadeh fuzzy AND operator. The new technique contains our original technique as a special case. Given the importance of PID control, we focus on Mamdani fuzzy PI and PD controllers in this paper and show in detail how to use the new technique for different configurations of the fuzzy PI/PD controllers. The controllers use two arbitrary trapezoidal fuzzy sets for each input variable, four arbitrary singleton output fuzzy sets, four fuzzy rules, Zadeh fuzzy AND operator, and the centroid defuzzifier. This configuration is more general and complicated than the Mamdani fuzzy PI/PD controllers in the current literature. It actually contains them as special cases. We call this configuration the generalized fuzzy PI/PD controller.

Keywords: Fuzzy control; PID control; Analytical structure

1. Introduction

It is well-documented that fuzzy control is effective in solving practical control problems (Yen, Langari, & Zadeh, 1995). Logically, many questions arise. Why is this the case? Is there anything special about the fuzzy controllers that make them well behave? Do the fuzzy controllers work well because their input–output structures are advantageously peculiar? What are their structures then? Other important questions include how to theoretically determine stability of fuzzy control systems, and how to design fuzzy control systems with as little trial-and-error effort as possible. More questions along these lines can be asked. After all, fuzzy control is nonlinear control and as such all the questions relevant to nonlinear control are applicable to fuzzy control (Ying, 2000).

The fundamental requirement for developing rigorous solutions to the above questions and the like is the availability of the analytical structure of the input–output relation of the fuzzy controllers. By "analytical structure" we mean the mathematical expression of a fuzzy controller that represents precisely the fuzzy controller without any approximation. Note that this is never an issue for conventional control because the analytical structure of a conventional controller, linear or nonlinear, is always readily available for analysis and design. Thus, the design goal is to design the controller structure and parameters on the basis of the given system model so that the resulting control system performance will meet user's performance specifications. For fuzzy control, in addition to this usual requirement, there exist few more major difficulties pertinent only to fuzzy control and irrelevant to conventional control. One of them is that the input–output structure of a fuzzy controller is usually mathematically unavailable after the controller is

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constructed. Indeed, most fuzzy controllers are constructed via so-called intelligent system approaches as opposed to the mathematical approaches exclusively used in conventional control. The fuzzy controllers have been treated and used as black-box controllers. Without the analytical structure information, precise and effective mathematical analysis and design are very difficult to achieve. Hence, the foremost issue is revealing the analytical structure of fuzzy controllers in such a way that the resulting structure is sensible in the context of conventional control theory. This is to say that merely deriving the structure is not useful enough and the structure must be represented in a form clearly understandable from control theory standpoint. Once the structure is well understood, analytical issues, including those listed above, can be explored using the well-developed conventional control theory (e.g., Ying, 2000; Ying, 1993a; Ying, 1994; Chen & Ying, 1997; Xu, Liu, & Hang, 1996; Malki, Li, & Chen, 1994).

Only two types of fuzzy AND operators are widely used in fuzzy control: product AND operator and Zadeh AND operator. Deriving the analytical structure for the fuzzy controllers that use the former operator is not difficult, regardless of the shape of input fuzzy sets. This is because it is rather straightforward to carry out multiplication of multiple membership functions in a fuzzy rule. However, revealing the analytical structure for the fuzzy controllers that use Zadeh fuzzy AND operator is far more difficult even for triangular fuzzy sets, which is the simplest fuzzy sets, because this operator requires the comparison of the membership functions (not numbers) and the use of the smallest function as the result of the AND operation.

In 1990, we developed the first technique in the literature for deriving the analytical structure of a fuzzy controller that uses the trapezoidal or triangular input fuzzy sets and Zadeh AND operator (Ying, Siler, & Buckley, 1990), and we showed specifically how to use the technique to derive the analytical structures of the fuzzy PI controllers. Although applicable to a wide range of fuzzy controllers, the technique is not general enough to be applicable to arbitrary trapezoidal/triangular fuzzy sets. It works when the input fuzzy sets are: (1) equally spaced trapezoidal/triangular fuzzy sets with the membership sum of two neighboring fuzzy sets always being equal to one (Fig. 1(a)), or (2) a class of equally spaced trapezoidal fuzzy sets with the

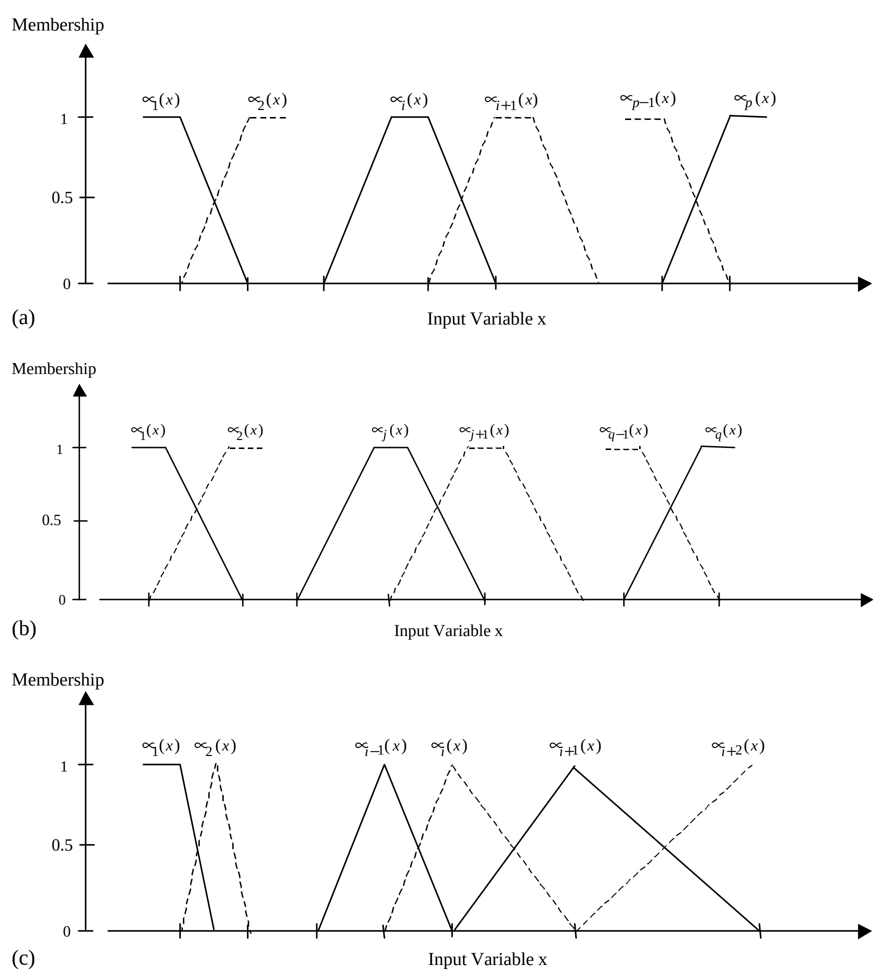

Fig. 1. Examples of trapezoidal/triangular input fuzzy sets that are solvable by the two existing structure-deriving techniques.

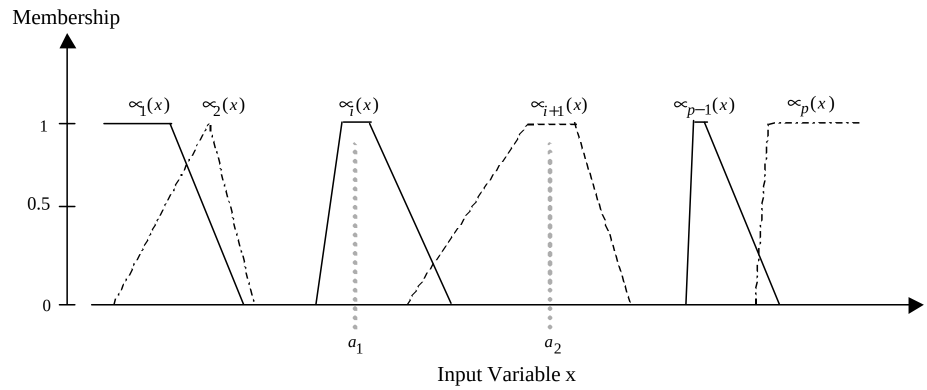

Fig. 2. Examples of trapezoidal/triangular input fuzzy sets unsolvable by the existing methods.

membership sum of two neighboring fuzzy sets not being equal to one (Fig. 1(b)). In either case, the shape and size of every set must be identical not only for each input variable but also for all input variables. Our method has been used by us as well as many other researchers for the structure analysis of various fuzzy controllers whose input fuzzy sets meet the above requirements (e.g., Ying, 1993a; Chen, Wang, Hsieh, & Chang, 1998; Hajjaji & Rachid, 1994; Li & Gatland, 1995; Li, 1998; Mann, Hu, & Gosine, 1999; Wong, Chou, & Mon, 1993). Based on this technique, Chen et al. developed a method that works for a special class of triangular fuzzy sets shown in Fig. 1(c) (Chen, Wang, Hsieh, & Chang, 1999). Compared with Figs. 1(a) and (b), the equal spacing requirement has been removed for the triangular fuzzy sets (but not for the trapezoidal fuzzy sets). The requirement on the membership sum being equal to one still remains. Up to date, these two techniques are the only ones available in the literature for the structure derivation.

Fuzzy controllers using arbitrary trapezoidal or triangular input fuzzy sets, such as those illustrated in Fig. 2, have been widely used in practice (Driankov et al., 1993; Passino & Yurkovich, 1998), probably more so than those using fuzzy sets highlighted in Fig. 1. Fuzzy sets shown in Fig. 2 are obviously more general than those shown in Fig. 1 for two reasons: (1) no two fuzzy sets are the same, and (2) the membership sum of two neighboring sets is not equal to one. Clearly, it is important to study the analytical structure of these more general and complicated fuzzy controllers. Unfortunately, there exists no knowledge in the current literature regarding the analytical structure of any fuzzy controller that uses Zadeh AND operator and the arbitrary trapezoidal/triangular input fuzzy sets. A structure-deriving technique substantially more general than the existing two must be developed first before such knowledge can be generated. No work in the literature has been found that addresses this significant but difficult problem.

In the present paper, we show how we have successfully resolved this challenge by developing a novel and general technique that contains the existing two techniques as special cases. While the new technique is universally applicable to any fuzzy controllers that use any number of the arbitrary trapezoidal/triangular input fuzzy sets for input variables, this paper will concentrate on a class of such fuzzy controllers for two good reasons. First, this class relates itself to PI or PD control. Given that PID control dominates 90% industrial process control worldwide (Astrom & Haagglund, 2001) studying fuzzy PID control is clearly important. Second, the fuzzy PI/PD controllers use two fuzzy sets for each input variable. This makes our presentation of the new technique more focused and consequently easier to understand. Nevertheless, to show the much broader applicability of our new technique, we devote one section (Section 4) to show, in relation to the fuzzy PI/PD controllers, the steps for deriving the analytical structure for the fuzzy controllers that use more than two fuzzy sets for each input variable. We also show some previous structure results obtained by us and some of other researchers are just special cases of our new structure results in Section 5.

2. Configuration of generalized Mamdani fuzzy PI controller

The generalized Mamdani fuzzy PI controller uses system output, $y(n)$, to calculate the scaled error and change of error of $y(n)$ at sampling time n:

$$E(n) = K_e e(n) = K_e(SP - y(n)),$$

$$R(n) = K_r r(n) = K_r(e(n) - e(n-1)),$$

where K_e and K_r are scaling factors, and SP is the output command signal. $E(n)$ and $R(n)$ are fuzzified by the arbitrary trapezoidal fuzzy sets shown in Fig. 3. Note that the triangular fuzzy sets are special cases of the trapezoidal ones when $P_1 = P_2 = Q_1 = Q_2 = 0$. The two fuzzy sets for $E(n)$ are denoted $\tilde{E}_1$ and $\tilde{E}_2$ whereas the two for $R(n)$ are designated as $\tilde{R}_1$ and $\tilde{R}_2$. Linguistic names may be assigned to these fuzzy sets. $\tilde{E}_1$ and $\tilde{R}_1$ may be called “Negative” whereas $\tilde{E}_2$ and $\tilde{R}_2$ “Positive”. The membership functions of the fuzzy sets are

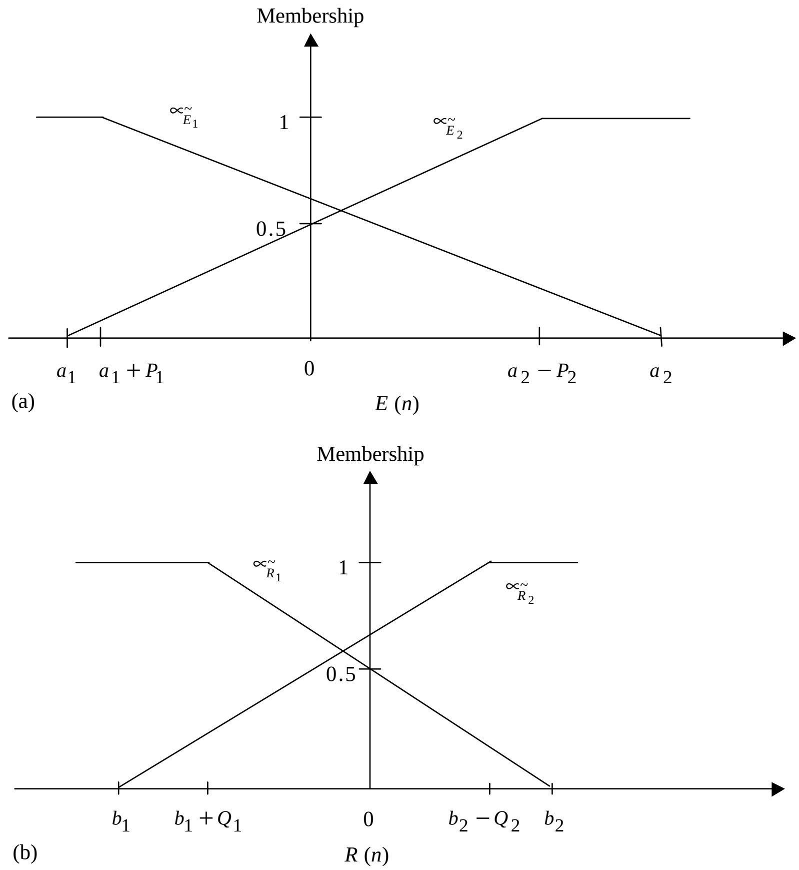

Fig. 3. A general configuration of input fuzzy sets: (a) two arbitrary trapezoidal fuzzy sets for $E(n)$, and (b) two arbitrary trapezoidal fuzzy sets for $R(n)$.

denoted as $\mu_{\tilde{E}_1}$, $\mu_{\tilde{E}_2}$, $\mu_{\tilde{R}_1}$ and $\mu_{\tilde{R}_2}$, respectively. It should be emphasized that the membership functions can be different from each other and we neither require $\mu_{\tilde{E}_1}+\mu_{\tilde{E}_2}=1$, nor $\mu_{\tilde{R}_1}+\mu_{\tilde{R}_2}=1$. These two equalities have been universal restrictions to all the fuzzy controllers whose analytical structures are reported in the literature (e.g., Ying et al., 1990; Ying, 1993a; Chen et al., 1998; Hajjaji & Rachid, 1994; Li & Gatland, 1995; Li, 1998; Mann et al., 1999; Wong et al., 1993) with notable exception of Ying (1999). The mathematical definitions of the fuzzy sets are as follows:

$$\mu_{\tilde{E}_2}=\begin{cases}0, & (-\infty,a_1],\\ \beta_2(E(n)-a_1), & [a_1,a_2-P_2],\\ 1, & [a_2-P_2,\infty),\end{cases}$$

$$\mu_{\tilde{E}_1}=\begin{cases}1, & (-\infty,a_1+P_1],\\ \beta_1(E(n)-a_2), & [a_1+P_1,a_2],\\ 0, & [a_2,\infty),\end{cases}$$

$$\mu_{\tilde{R}_2}=\begin{cases}0, & (-\infty,b_1],\\ \theta_2(R(n)-b_1), & [b_1,b_2-Q_2],\\ 1, & [b_2-Q_2,\infty),\end{cases}$$

$$\mu_{\tilde{R}_1}=\begin{cases}1, & (-\infty,b_1+Q_1],\\ \theta_1(R(n)-b_2), & [b_1+Q_1,b_2],\\ 0, & [b_2,\infty),\end{cases}$$

where

$$\beta_2=\frac{1}{a_2-a_1-P_2},\qquad \beta_1=-\frac{1}{a_2-a_1-P_1},$$
$$\theta_2=\frac{1}{b_2-b_1-Q_2},\qquad \theta_1=-\frac{1}{b_2-b_1-Q_1}.$$

These four parameters are the slopes of the linear segments of the four input fuzzy sets. Note that β_2 and θ_2 are positive and β_1 and θ_1 are negative.

Four fuzzy rules are used:

IF $E(n)$ is $\tilde{E}_2$ AND $R(n)$ is $\tilde{R}_2$ THEN $\Delta u(n)$ is $\tilde{H}_1$ (Rule 1)

IF $E(n)$ is $\tilde{E}_2$ AND $R(n)$ is $\tilde{R}_1$ THEN $\Delta u(n)$ is $\tilde{H}_2$ (Rule 2)

IF $E(n)$ is $\tilde{E}_1$ AND $R(n)$ is $\tilde{R}_2$ THEN $\Delta u(n)$ is $\tilde{H}_3$ (Rule 3)

IF $E(n)$ is $\tilde{E}_1$ AND $R(n)$ is $\tilde{R}_1$ THEN $\Delta u(n)$ is $\tilde{H}_4$ (Rule 4)

where $\Delta u(n)$ is the change of controller output and $\tilde{H}_i$, $i=1,2,3,4$, is a singleton output fuzzy set whose membership value is nonzero only at $\Delta u(n)=h_i$. Unlike the previous studies (e.g., Ying et al., 1990; Ying, 1993a; Malki et al., 1994) in which $h_1=-h_4$ and $h_2=h_3=0$, we place no restriction on the value of h_i in this study. A number of different inference methods, such as Mamdani minimum inference method, can be used for the fuzzy inference in the rules (Ying, 1993a). Their inference results will be the same because the output fuzzy sets are of singleton type. Zadeh fuzzy logic AND operator is used to realize the AND operations in the rules. Denoting μ_i the membership value that resulted from Zadeh AND operation in Rule i:

$$\mu_1 = \min(\mu_{\tilde{E}_2}, \mu_{\tilde{R}_2}), \quad \mu_2 = \min(\mu_{\tilde{E}_2}, \mu_{\tilde{R}_1}),$$

$$\mu_3 = \min(\mu_{\tilde{E}_1}, \mu_{\tilde{R}_2}), \quad \mu_4 = \min(\mu_{\tilde{E}_1}, \mu_{\tilde{R}_1}),$$

where min( ) is the operation taking the lesser of the two membership values as the result. The popular centroid defuzzifier is employed, which yields

$$\Delta U(n) = K_{\Delta u}\frac{\mu_1 h_1 + \mu_2 h_2 + \mu_3 h_3 + \mu_4 h_4}{\mu_1 + \mu_2 + \mu_3 + \mu_4},$$

where $K_{\Delta u}$ is a scaling factor. The output of the fuzzy controller at sampling time n is

$$U(n) = U(n-1) + \Delta U(n).$$

We should point out that if either input variable is fuzzified by more than two fuzzy sets, the resulting analytical structure will not be of PID type (e.g., Ying, 1993b; Ying, 1993c; Ying, 1999). Thus, the above configuration is the most general fuzzy PI controller. We call it the generalized Mamdani fuzzy PI controller.

3. General structure-deriving technique and its demonstrative derivation results involving the generalized fuzzy PI controller

We now present the new and general technique through the example of deriving the analytical structure of the generalized fuzzy PI controller under different settings of the fuzzy sets and rules. Derivation for the fuzzy controller when $E(n)$ is outside $[a_1, a_2]$ and $R(n)$ is outside $[b_1, b_2]$ can be achieved using our original technique developed before. The result will be the same as the fuzzy controllers we studied before Ying (1993a). For brevity, we will not show the result, nor will we show the result when either $E(n)$ is outside $[a_1, a_2]$ or $R(n)$ is outside $[b_1, b_2]$ (but not when both are outside the intervals).

3.1. General case when both trapezoidal input fuzzy sets and fuzzy rules are arbitrary

We now consider a most general case when the input fuzzy sets are general (i.e., little restrictive), as shown in Fig. 3, and the fuzzy rules are general in that the rule consequent are arbitrary (i.e., h_1–h_4 can be any values). There are four fuzzy sets; they all differ from each other. The fuzzy sets are characterized by the following relationships: $a_2 - a_1 > b_2 - b_1$, $P_2 > P_1$, $P_2 > Q_1 > Q_2 > P_1$, and $a_2 - P_2 - a_1 - P_1 > b_2 - Q_2 - b_1 - Q_1$.

One key step for a successful derivation is to divide the input space $[a_1, a_2] \times [b_1, b_2]$ into a number of regions so that in each region, a unique inequality relationship can be obtained for each fuzzy rule between the two membership functions being ANDed by Zadeh AND operator. The division method that we previously introduced in Ying et al. (1990) works only for fuzzy sets similar to those shown in Figs. 1(a) and (b). It is not applicable to the fuzzy controllers in this paper because the input fuzzy sets are arbitrary and no longer satisfy the requirements mentioned in the Introduction. To this end, a novel and more general input space division technique is needed. We have developed it as described below.

Fig. 4 shows the input space division result specifically for the general fuzzy PI controller when it uses the fuzzy sets shown in Fig. 3. Unlike the previous cases where one input space division can be used for all the fuzzy rules (Ying et al., 1990), each fuzzy rule must now have its own individual input space division due to the arbitrariness of the fuzzy sets. Thus, there are four different input space divisions for the four rules (Figs. 4(a)–(d)). Note that there are five regions for each input space division. We called each region an Input Combination (IC) (Ying et al., 1990; Ying, 1993a). The regions are labeled A_1–A_5 for Rule 1, B_1–B_5 for Rule 2, C_1–C_5 for Rule 3, and D_1–D_5 for Rule 4. The divisions depend on the magnitudes of P_1, P_2, Q_1, and Q_2. The divisions shown in Fig. 4 are valid only when the above-mentioned five relationships of the input fuzzy sets hold. The divisions will look different if the relationships change. For instance, if $P_2 = 0$, regions A_2, A_5, B_3, and B_5 will not exist any more.

The corresponding result of the fuzzy AND operation for each region is also given in Fig. 4, side by side with its IC number. For instance, the result for region A_4 is $\mu_{\tilde{R}_2}$ (Fig. 4(a)). To understand how such a result is obtained, consider the following. In this region, as far as Rule 1 is concerned, in the rectangular area $[a_1, a_2 - P_2] \times [b_1, b_2 - Q_2]$, $\mu_{\tilde{E}_2}$ for $E(n)$ is always larger than $\mu_{\tilde{R}_2}$ for $R(n)$. The two membership functions become equal on the line connecting (a_1, b_1) and $(a_2 - P_2, b_2 - Q_2)$. It is this line that divides the rectangular area into two regions: A_3 and A_4. Hence, the result of Zadeh AND operation should be $\mu_{\tilde{R}_2}$. All the membership functions in Fig. 4 are obtained in this way.

These divisions are for the individual fuzzy rules only. Nevertheless, all four rules must be simultaneously considered to derive the controller structure. This is because all the rules are involved in calculating controller output at every sampling time. This simultaneous consideration is achieved if we superimpose the four individual input space divisions shown in Fig. 4 to form an overall input space division for all the rules (Fig. 5). It turns out that there are a total of 26 regions (i.e., IC1–IC26). Note that the number and shape of these regions depend on the four individual input space

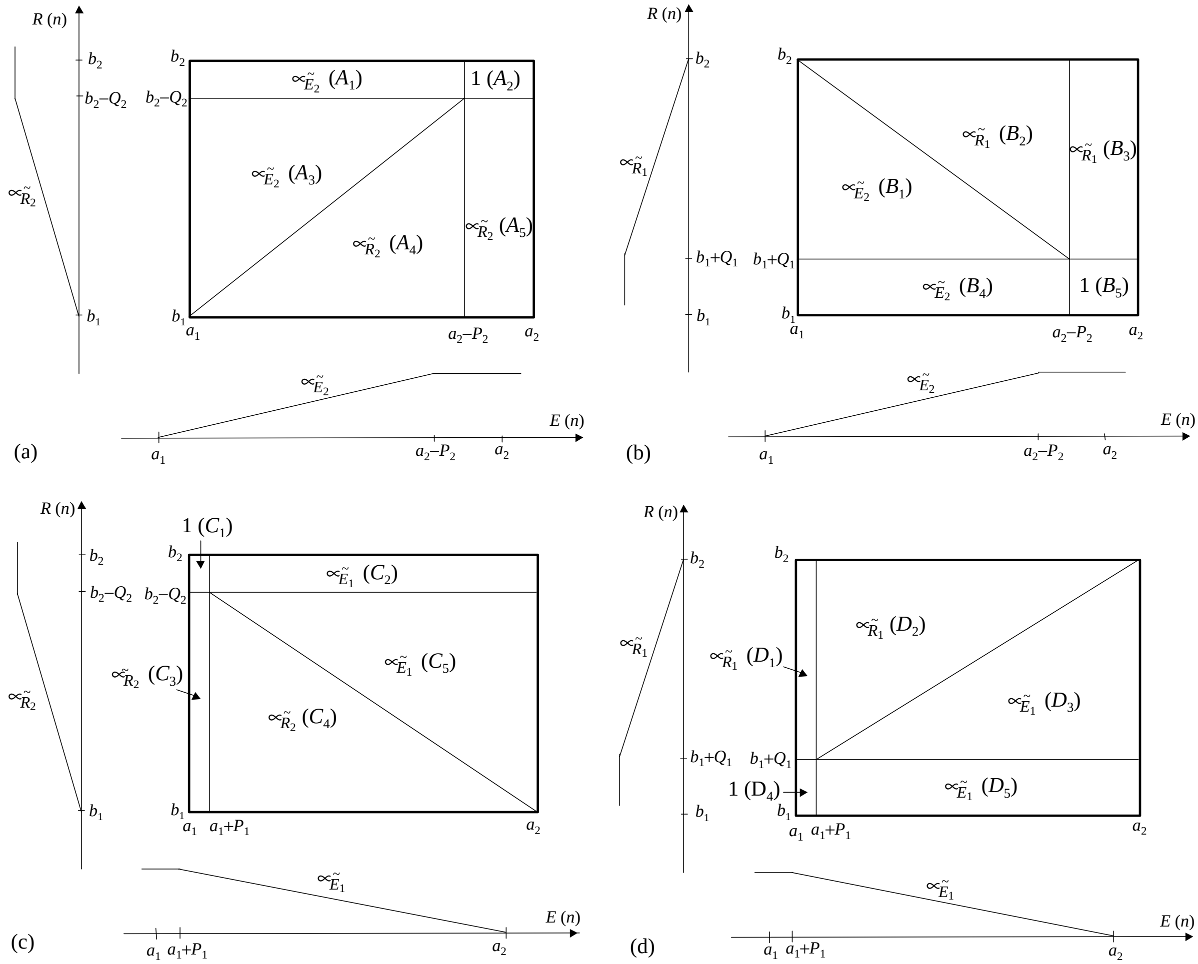

Fig. 4. Division of input space $[a_1, a_2] \times [b_1, b_2]$ for evaluation of Zadeh fuzzy AND operation: (a) for Rule 1, (b) for Rule 2, (c) for Rule 3, and (d) for Rule 4.

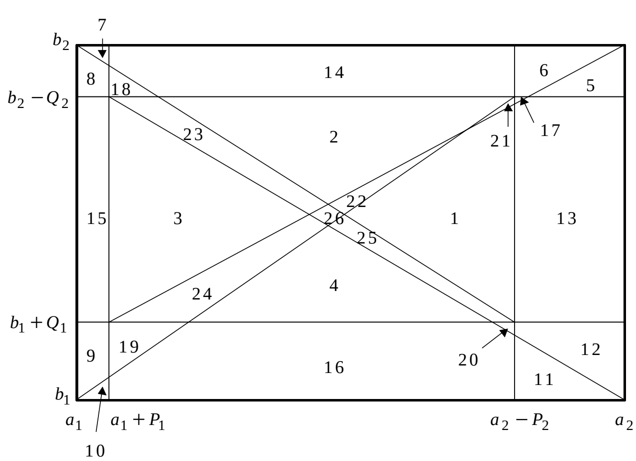

Fig. 5. Superimpose the divisions in Figs. 4(a)–(d) to form an overall division for evaluating Zadeh fuzzy AND operation for the four fuzzy rules simultaneously.

divisions shown in Fig. 4, which in turn, as we have discussed above, depend on the input fuzzy sets (i.e., the values of a_1, a_2, b_1, b_2, P_1, P_2, Q_1, and Q_2). Regardless of the values of these parameters, the largest possible number of regions is 26.

Now, for each region in Fig. 5, we can decide, using Figs. 4 and 5, the resulting membership function for each fuzzy rule due to Zadeh fuzzy AND operation. For example, IC2 (i.e., the region labeled 2 in Fig. 5) corresponds to A_3 for Rule 1, to B_2 for Rule 2, to C_5 for Rule 3, and to D_2 for Rule 4. According to Fig. 4, the membership functions that resulted from Zadeh AND operation for these regions are: $\mu_{\tilde{E}_2}$ for A_3, $\mu_{\tilde{R}_1}$ for B_2, $\mu_{\tilde{E}_1}$ for C_5, and $\mu_{\tilde{R}_1}$ for D_2. This process is repeated for IC1–IC26 and the complete result is listed in Table 1. We emphasize that this list holds true only for the division given in Fig. 5. It can be different if the input fuzzy sets are different in terms of the relationships among $a_1, a_2, b_1, b_2, P_1, P_2, Q_1$, and Q_2.

Putting these membership functions (note that they are functions, not numbers) into the defuzzifier, we will get the analytical structure of the fuzzy controller after some algebraic manipulations. Continuing the IC2 example above,

Table 1
Result of Zadeh AND operation in each fuzzy rule for the overall input space division shown in Fig. 5

IC No.	Rule 1	Rule 2	Rule 3	Rule 4
1	$\mu_{\tilde{R}_2}$ (A_4)	$\mu_{\tilde{R}_1}$ (B_2)	$\mu_{\tilde{E}_1}$ (C_5)	$\mu_{\tilde{E}_1}$ (D_3)
2	$\mu_{\tilde{E}_2}$ (A_3)	$\mu_{\tilde{R}_1}$ (B_2)	$\mu_{\tilde{E}_1}$ (C_5)	$\mu_{\tilde{R}_1}$ (D_2)
3	$\mu_{\tilde{E}_2}$ (A_3)	$\mu_{\tilde{E}_2}$ (B_1)	$\mu_{\tilde{R}_2}$ (C_4)	$\mu_{\tilde{R}_1}$ (D_2)
4	$\mu_{\tilde{R}_2}$ (A_4)	$\mu_{\tilde{E}_2}$ (B_1)	$\mu_{\tilde{R}_2}$ (C_4)	$\mu_{\tilde{E}_1}$ (D_3)
5	1 (A_2)	$\mu_{\tilde{R}_1}$ (B_3)	$\mu_{\tilde{E}_1}$ (C_2)	$\mu_{\tilde{E}_1}$ (D_3)
6	1 (A_2)	$\mu_{\tilde{R}_1}$ (B_3)	$\mu_{\tilde{E}_1}$ (C_2)	$\mu_{\tilde{R}_1}$ (D_2)
7	$\mu_{\tilde{E}_2}$ (A_1)	$\mu_{\tilde{R}_1}$ (B_2)	1 (C_1)	$\mu_{\tilde{R}_1}$ (D_1)
8	$\mu_{\tilde{E}_2}$ (A_1)	$\mu_{\tilde{E}_2}$ (B_1)	1 (C_1)	$\mu_{\tilde{R}_1}$ (D_1)
9	$\mu_{\tilde{E}_2}$ (A_3)	$\mu_{\tilde{E}_2}$ (B_4)	$\mu_{\tilde{R}_2}$ (C_3)	1 (D4)
10	$\mu_{\tilde{R}_2}$ (A4)	$\mu_{\tilde{E}_2}$ (B4)	$\mu_{\tilde{R}_2}$ (C3)	1 (D_4)
11	$\mu_{\tilde{R}_2}$ (A_5)	1 (B_5)	$\mu_{\tilde{R}_2}$ (C_4)	$\mu_{\tilde{E}_1}$ (D_5)
12	$\mu_{\tilde{R}_2}$ (A_5)	1 (B_5)	$\mu_{\tilde{E}_1}$ (C_5)	$\mu_{\tilde{E}_1}$ (D_5)
13	$\mu_{\tilde{R}_2}$ (A_5)	$\mu_{\tilde{R}_1}$ (B_3)	$\mu_{\tilde{E}_1}$ (C_5)	$\mu_{\tilde{E}_1}$ (D_3)
14	$\mu_{\tilde{E}_2}$ (A_1)	$\mu_{\tilde{R}_1}$ (B_2)	$\mu_{\tilde{E}_1}$ (C_2)	$\mu_{\tilde{R}_1}$ (D_2)
15	$\mu_{\tilde{E}_2}$ (A_3)	$\mu_{\tilde{E}_2}$ (B_1)	$\mu_{\tilde{R}_2}$ (C_3)	$\mu_{\tilde{R}_1}$ (D_1)
16	$\mu_{\tilde{R}_2}$ (A_4)	$\mu_{\tilde{E}_2}$ (B_4)	$\mu_{\tilde{R}_2}$ (C_4)	$\mu_{\tilde{E}_1}$ (D_5)
17	$\mu_{\tilde{R}_2}$ (A_5)	$\mu_{\tilde{R}_1}$ (B_3)	$\mu_{\tilde{E}_1}$ (C_5)	$\mu_{\tilde{R}_1}$ (D_2)
18	$\mu_{\tilde{E}_2}$ (A_1)	$\mu_{\tilde{E}_2}$ (B_1)	$\mu_{\tilde{E}_1}$ (C_2)	$\mu_{\tilde{R}_1}$ (D_2)
19	$\mu_{\tilde{E}_2}$ (A_3)	$\mu_{\tilde{E}_2}$ (B_4)	$\mu_{\tilde{R}_2}$ (C_4)	$\mu_{\tilde{E}_1}$ (D_5)
20	$\mu_{\tilde{R}_2}$ (A_4)	$\mu_{\tilde{E}_2}$ (B_4)	$\mu_{\tilde{E}_1}$ (C_5)	$\mu_{\tilde{E}_1}$ (D_5)
21	$\mu_{\tilde{R}_2}$ (A_4)	$\mu_{\tilde{R}_1}$ (B_2)	$\mu_{\tilde{E}_1}$ (C_5)	$\mu_{\tilde{R}_1}$ (D_2)
22	$\mu_{\tilde{E}_2}$ (A_3)	$\mu_{\tilde{R}_1}$ (B_2)	$\mu_{\tilde{E}_1}$ (C_5)	$\mu_{\tilde{E}_1}$ (D_3)
23	$\mu_{\tilde{E}_2}$ (A_3)	$\mu_{\tilde{E}_2}$ (B_1)	$\mu_{\tilde{R}_2}$ (C_5)	$\mu_{\tilde{R}_1}$ (D_2)
24	$\mu_{\tilde{E}_2}$ (A_3)	$\mu_{\tilde{E}_2}$ (B_1)	$\mu_{\tilde{R}_2}$ (C_4)	$\mu_{\tilde{E}_1}$ (D_3)
25	$\mu_{\tilde{R}_2}$ (A_4)	$\mu_{\tilde{E}_2}$ (B_1)	$\mu_{\tilde{E}_1}$ (C_5)	$\mu_{\tilde{R}_1}$ (D_3)
26	$\mu_{\tilde{E}_2}$ (A_3)	$\mu_{\tilde{E}_2}$ (B_1)	$\mu_{\tilde{E}_1}$ (C_5)	$\mu_{\tilde{E}_1}$ (D_3)

the analytical structure in this region is

$$\Delta U(n) = \frac{\mu_1 h_1 + \mu_2 h_2 + \mu_3 h_3 + \mu_4 h_4}{\mu_1 + \mu_2 + \mu_3 + \mu_4}$$

$$= \frac{\mu_{\tilde{E}_2} h_1 + \mu_{\tilde{R}_1} h_2 + \mu_{\tilde{E}_1} h_3 + \mu_{\tilde{R}_1} h_4}{\mu_{\tilde{E}_2} + \mu_{\tilde{R}_1} + \mu_{\tilde{E}_1} + \mu_{\tilde{R}_1}}$$

$$= K_{\Delta u} \frac{(h_1\beta_2 + h_3\beta_1)K_e e(n) + (h_2\theta_1 + h_4\theta_1)K_r r(n) + (-h_1 a_1 \beta_2 - h_2 b_2 \theta_1 - h_3 a_2 \beta_1 - h_4 b_2 \theta_1)}{(\beta_1 + \beta_2)K_e e(n) + 2\theta_1 K_r r(n) - a_1\beta_2 - a_2\beta_1 - 2b_2\theta_1}.$$

We have obtained the analytical structure for all the regions, from IC1 to IC26, as illustrated in Table 2. We should point out that the control action formula in IC1 and IC13, IC2 and IC14, IC3 and IC15, and IC4 and IC16 are, respectively, exactly the same. Also, it should be noticed that the control action changes continuously and smoothly across the boundary of any adjacent regions. On any boundary line or at any conjunction point of multiple regions, the control action can be computed using any of the control algorithms involved. The result will be the same.

In every IC, the controller structure can be expressed as

$$\Delta U(n) = \frac{C_1 e(n) + C_2 r(n) + C_3}{D_1 e(n) + D_2 r(n) + D_3},$$

where C_i and D_i, $i = 1, 2, 3$, are constants. In different ICs, these constants may be different. Using IC2 as an example, $C_1 = (h_1\beta_2 + h_3\beta_1)K_e$, $C_2 = (h_2\theta_1 + h_4\theta_1)K_r$, $C_3 = -h_1 a_1 \beta_2 - h_2 b_2 \theta_1 - h_3 a_2 \beta_1 - h_4 b_2 \theta_1$, $D_1 = (\beta_1 + \beta_2)K_e$, $D_2 = 2\theta_1 K_r$, and $D_3 = -a_1\beta_2 - a_2\theta_1 - 2b_2\theta_1$.

If we designate

$$K_i(e,r) = \frac{C_1}{D_1 e(n) + D_2 r(n) + D_3},$$

$$K_p(e,r) = \frac{C_2}{D_1 e(n) + D_2 r(n) + D_3},$$

$$\delta(e,r) = \frac{C_3}{D_1 e(n) + D_2 r(n) + D_3},$$

then the structure in each of the 26 regions is a nonlinear PI controller with variable proportional-gain $K_p(e,r)$, variable integral-gain $K_i(e,r)$, and variable control offset $\delta(e,r)$:

$$\Delta U(n) = K_i(e,r)e(n) + K_p(e,r)r(n) + \delta(e,r).$$

Table 2
Analytical structure of the generalized fuzzy PI controller for the overall input space division shown in Fig. 5

IC No.	$\Delta U(n)=$
1	$K_{\Delta u}\frac{(h_3+h_4)\beta_1K_ee(n)+(h_1\theta_2+h_2\theta_1)K_rr(n)+(-h_1b_1\theta_2-h_2b_2\theta_1-h_3a_2\beta_1-h_4a_2\beta_1)}{2\beta_1K_ee(n)+(\theta_1+\theta_2)K_rr(n)-2a_2\beta_1-b_1\theta_2-b_2\theta_1}$
2	$K_{\Delta u}\frac{(h_1\beta_2+h_3\beta_1)K_ee(n)+(h_2\theta_1+h_4\theta_1)K_rr(n)+(-h_1a_1\beta_2-h_2b_2\theta_1-h_3a_2\beta_1-h_4b_2\theta_1)}{(\beta_1+\beta_2)K_ee(n)+2\theta_1K_rr(n)-a_1\beta_2-a_2\beta_1-2b_2\theta_1}$
3	$K_{\Delta u}\frac{(h_1+h_2)\beta_2K_ee(n)+(h_3\theta_2+h_4\theta_1)K_rr(n)+(-h_1a_1\beta_2-h_2a_1\beta_2-h_3b_1\theta_2-h_4b_2\theta_1)}{2\beta_2K_ee(n)+(\theta_1+\theta_2)K_rr(n)-2a_1\beta_2-b_1\theta_2-b_2\theta_1}$
4	$K_{\Delta u}\frac{(h_2\beta_2+h_4\beta_1)K_ee(n)+(h_1+h_3)\theta_2K_rr(n)+(-h_1b_1\theta_2-h_2a_1\beta_2-h_3b_1\theta_2-h_4a_2\beta_1)}{(\beta_1+\beta_2)K_ee(n)+2\theta_2K_rr(n)-a_1\beta_2-a_2\beta_1-2b_1\theta_2}$
5	$K_{\Delta u}\frac{(h_3+h_4)\beta_1K_ee(n)+h_2\theta_1K_rr(n)+(h_1-h_2b_2\theta_1-h_3a_2\beta_1-h_4a_2\beta_1)}{2\beta_1K_ee(n)+\theta_1K_rr(n)-2a_2\beta_1-b_2\theta_1+1}$
6	$K_{\Delta u}\frac{h_3\beta_1K_ee(n)+(h_2+h_4)\theta_1K_rr(n)+(h_1-h_2b_2\theta_1-h_3a_2\beta_1-h_4b_2\theta_1)}{\beta_1K_ee(n)+2\theta_1K_rr(n)-a_2\beta_1-2b_2\theta_1+1}$
7	$K_{\Delta u}\frac{h_1\beta_2K_ee(n)+(h_2+h_4)\theta_1K_rr(n)+(-h_1a_1\beta_2-h_2b_2\theta_1+h_3-h_4b_2\theta_1)}{\beta_2K_ee(n)+2\theta_1K_rr(n)-a_1\beta_2-2b_2\theta_1+1}$
8	$K_{\Delta u}\frac{(h_1+h_2)\beta_2K_ee(n)+h_4\theta_1K_rr(n)+(-h_1a_1\beta_2-h_2a_1\beta_2+h_3-h_4b_2\theta_1)}{2\beta_2K_ee(n)+\theta_1K_rr(n)-2a_1\beta_2-b_2\theta_1+1}$
9	$K_{\Delta u}\frac{(h_1+h_2)\beta_2K_ee(n)+h_3\theta_2K_rr(n)+(-h_1a_1\beta_2-h_2a_1\beta_2-h_3b_1\theta_2+h_4)}{2\beta_2K_ee(n)+\theta_2K_rr(n)-2a_1\beta_2-b_1\theta_2+1}$
10	$K_{\Delta u}\frac{h_2\beta_2K_ee(n)+(h_1+h_3)\theta_2K_rr(n)+(-h_1b_1\theta_2-h_2a_1\beta_2-h_3b_1\theta_2+h_4)}{\beta_2K_ee(n)+2\theta_2K_rr(n)-a_1\beta_2-2b_1\theta_2+1}$
11	$K_{\Delta u}\frac{h_4\beta_1K_ee(n)+(h_1+h_3)\theta_2K_rr(n)+(-h_1b_1\theta_2+h_2-h_3b_1\theta_2-h_4a_2\beta_1)}{\beta_1K_ee(n)+2\theta_2K_rr(n)-a_2\beta_1-2b_1\theta_2+1}$
12	$K_{\Delta u}\frac{(h_3+h_4)\beta_1K_ee(n)+h_1\theta_2K_rr(n)+(-h_1b_1\theta_2+h_2-h_3a_2\beta_1-h_4a_2\beta_1)}{2\beta_1K_ee(n)+\theta_2K_rr(n)-2a_2\beta_1-b_1\theta_2+1}$
13	$K_{\Delta u}\frac{(h_3+h_4)\beta_1K_ee(n)+(h_1\theta_2+h_2\theta_1)K_rr(n)+(-h_1b_1\theta_2-h_2b_2\theta_1-h_3a_2\beta_1-h_4a_2\beta_1)}{2\beta_1K_ee(n)+(\theta_1+\theta_2)K_rr(n)-2a_2\beta_1-b_1\theta_2-b_2\theta_1}$
14	$K_{\Delta u}\frac{(h_1\beta_2+h_3\beta_1)K_ee(n)+(h_2+h_4)\theta_1K_rr(n)+(-h_1a_1\beta_2-h_2b_2\theta_1-h_3a_2\beta_1-h_4b_2\theta_1)}{(\beta_1+\beta_2)K_ee(n)+2\theta_1K_rr(n)-a_1\beta_2-a_2\beta_1-2b_2\theta_1}$
15	$K_{\Delta u}\frac{(h_1+h_2)\beta_2K_ee(n)+(h_3\theta_2+h_4\theta_1)K_rr(n)+(-h_1a_1\beta_2-h_2a_1\beta_2-h_3b_1\theta_2-h_4b_2\theta_1)}{2\beta_2K_ee(n)+(\theta_1+\theta_2)K_rr(n)-2a_1\beta_2-b_1\theta_2-b_2\theta_1}$
16	$K_{\Delta u}\frac{(h_2\beta_2+h_4\beta_1)K_ee(n)+(h_1+h_3)\theta_2K_rr(n)+(-h_1b_1\theta_2-h_2a_1\beta_2-h_3b_1\theta_2-h_4a_2\beta_1)}{(\beta_1+\beta_2)K_ee(n)+2\theta_2K_rr(n)-a_1\beta_2-a_2\beta_1-2b_1\theta_2}$
17	$K_{\Delta u}\frac{h_3\beta_1K_ee(n)+(h_1\theta_2+h_2\theta_1+h_4\theta_2)K_rr(n)+(-h_1b_1\theta_2-h_2b_2\theta_1-h_3a_2\beta_1-h_4b_2\theta_1)}{\beta_1K_ee(n)+(2\theta_1+\theta_2)K_rr(n)-a_2\beta_1-b_1\theta_2-2b_2\theta_1}$
18	$K_{\Delta u}\frac{(h_1\beta_2+h_2\beta_2+h_3\beta_1)K_ee(n)+h_4\theta_1K_rr(n)+(-h_1a_1\beta_2-h_2a_1\beta_2-h_3a_2\beta_1-h_4b_2\theta_1)}{(\beta_1+2\beta_2)K_ee(n)+\theta_1K_rr(n)-2a_1\beta_2-a_2\beta_1-b_2\theta_1}$
19	$K_{\Delta u}\frac{(h_1\beta_2+h_2\beta_2+h_4\beta_1)K_ee(n)+h_3\theta_2K_rr(n)+(-h_1a_1\beta_2-h_2a_1\beta_2-h_3b_1\theta_2-h_4a_2\beta_1)}{(\beta_1+2\beta_2)K_ee(n)+\theta_2K_rr(n)-2a_1\beta_2-a_2\beta_1-b_1\theta_2}$
20	$K_{\Delta u}\frac{(h_2\beta_2+h_3\beta_1+h_4\beta_1)K_ee(n)+h_1\theta_2K_rr(n)+(-h_1b_1\theta_2-h_2a_1\beta_2-h_3a_2\beta_1-h_4a_2\beta_1)}{(2\beta_1+\beta_2)K_ee(n)+\theta_2K_rr(n)-a_1\beta_2-2a_2\beta_1-b_1\theta_2}$
21	$K_{\Delta u}\frac{h_3\beta_1K_ee(n)+(h_1\theta_2+h_2\theta_1+h_4\theta_1)K_rr(n)+(-h_1b_1\theta_2-h_2b_2\theta_1-h_3a_2\beta_1-h_4b_2\theta_1)}{\beta_1K_ee(n)+(2\theta_1+\theta_2)K_rr(n)-a_2\beta_1-b_1\theta_2-2b_2\theta_1}$
22	$K_{\Delta u}\frac{(h_1\beta_2+h_3\beta_1+h_4\beta_1)K_ee(n)+h_2\theta_1K_rr(n)+(-h_1a_1\beta_2-h_2b_2\theta_1-h_3a_2\beta_1-h_4a_2\beta_1)}{(2\beta_1+\beta_2)K_ee(n)+\theta_1K_rr(n)-a_1\beta_2-2a_2\beta_1-b_2\theta_1}$
23	$K_{\Delta u}\frac{(h_1\beta_2+h_2\beta_2+h_3\beta_1)K_ee(n)+h_4\theta_1K_rr(n)+(-h_1a_1\beta_2-h_2a_1\beta_2-h_3a_2\beta_1-h_4b_2\theta_1)}{(\beta_1+2\beta_2)K_ee(n)+\theta_1K_rr(n)-2a_1\beta_2-a_2\beta_1-b_2\theta_1-b_2\theta_1}$

Table 2 (*continuted.*)

IC No.	$\Delta U(n)=$
24	$K_{\Delta u}\dfrac{(h_1\beta_2+h_2\beta_2+h_4\beta_1)K_e e(n)+h_3\theta_2K_r r(n)+(-h_1a_1\beta_2-h_2a_1\beta_2-h_3b_1\theta_2-h_4a_2\beta_1)}{(\beta_1+2\beta_2)K_e e(n)+\theta_2K_r r(n)-2a_1\beta_2-a_2\beta_1-b_1\theta_2}$
25	$K_{\Delta u}\dfrac{(h_2\beta_2+h_3\beta_1+h_4\beta_1)K_e e(n)+h_1\theta_2K_r r(n)+(-h_1b_1\theta_2-h_2a_1\beta_2-h_3a_2\beta_1-h_4a_2\beta_1)}{(2\beta_1+\beta_2)K_e e(n)+\theta_2K_r r(n)-a_1\beta_2-2a_2\beta_1-b_1\theta_2}$
26	$K_{\Delta u}\dfrac{(h_1\beta_2+h_2\beta_2+h_3\beta_1+h_4\beta_1)K_e e(n)+(-h_1a_1\beta_2-h_2a_1\beta_2-h_3a_2\beta_1-h_4a_2\beta_1)}{2(\beta_1+\beta_2)K_e e(n)-2a_1\beta_2-2a_2\beta_1}$

The gains and offset vary with change of $e(n)$ and $r(n)$. The only exception is IC26 whose $K_p(e,r)=0$, which leads to a nonlinear I controller with variable gain and control offset. It is viewed as a special case of nonlinear PI controller.

When $e(n)=r(n)=0$, $\Delta U(n)=\delta(0,0)$. For most control applications, it is desired that $\delta(0,0)=0$ while in other situations this may not be the case. In either case, the components of the fuzzy controller should be properly chosen to realize the desired goal.

The idea of dividing input space into IC regions was first developed by the present author in 1987 (Ying, 1987) (see also Ying et al. (1990)). The purpose was to derive the analytical structure of the fuzzy controllers. It is interesting to point out that independently in the field of conventional control some of the recent controller design methods also use the concept of the input signal state space division (e.g., Bemporad, Morari, Dua, & Pistikopoulos, 2002; Da Dona, Goodwin, & Seron, 2002). Such a connection between the fuzzy controllers and the "mainstream" controllers is important as it provides opportunities to explore, for example, the possibility of applying the conventional control design schemes to the design of the fuzzy controllers.

We summarize the above results in the form of theorem.

Theorem. *The generalized fuzzy PI controller is a nonlinear PI controller with variable gain and variable control offset.*

3.2. General case when trapezoidal input fuzzy sets and fuzzy rules are arbitrary but symmetric

Let us now derive the analytical structure of the fuzzy controller that uses the symmetric input fuzzy sets shown in Fig. 6. In comparison with Fig. 3: $a_1=-a_2=L_1$, $b_1=-b_2=L_2$, $L_1>L_2, P_1=P_2=P, Q_1=Q_2=Q$, $P>Q$, and $L_1-P>L_2-Q$. Without loss of generality, we assume that $h_1=-h_4=H_1$ and $h_2=-h_3=H_2$ where H_1 is positive, H_2 is not negative (it may be zero), and $H_1>H_2$ (intuitively Rule 1 or Rule 4 should produce larger control action, in an absolute value sense, than Rule 2 or Rule 3 does). Keeping in mind the above relationships, the four input space divisions shown in Fig. 4 can still be directly used. Then, superimpose them to form an overall division for all the rules (Fig. 7). This time, there are a total of 25 ICs. The shape and location for some of the ICs are different than those in Fig. 5. For instance, IC21 shown in Fig. 5 no longer exists, nor do IC18, IC19, and IC20. Using the procedure described above, we can first obtain the result of Zadeh fuzzy AND operation (Table 3) and can then derive the analytical structure (Table 4). Obviously, Our theorem holds for this configuration of the generalized fuzzy PI controller as well.

3.3. Analytical structure for corresponding generalized fuzzy PD controllers

All the above results can be easily converted to become the analytical structure for the corresponding generalized fuzzy PD controller after only two simple modifications. First, change $\Delta u(n)$ to $u(n)$ in all the fuzzy rule consequent, $u(n)$ being the controller output. Second, replace $\Delta U(n)$ by $U(n)$ in Tables 2, 4 and 5. The resulting expressions describe the analytical structure of the generalized fuzzy PD controller under different settings of the fuzzy sets and fuzzy rules.

We now show why this is the case. A discrete-time PID controller in position form is described by

$$U(n)=K_pe(n)+K_i\sum_{i=0}^{n}e(i)+K_dr(n),$$

where K_p, K_i and K_d are proportional-gain, integral-gain and derivative-gain, respectively.

The incremental form of the PID controller is

$$\Delta U(n)=U(n)-U(n-1)=K_pr(n)+K_ie(n)+K_dd(n),$$

where $d(n)=r(n)-r(n-1)$. When K_d is set to zero, the PID controller becomes a PI controller in incremental form

$$\Delta U(n)=K_ie(n)+K_pr(n),$$

whereas when $K_i=0$, the PID controller in position form reduces to a PD controller in position form

$$U(n)=K_pe(n)+K_dr(n).$$

One sees that the PD controller in position form becomes the PI controller in incremental form if (1) $e(n)$ and $r(n)$ exchange positions, (2) K_d is replaced by K_i, and (3) $\Delta U(n)$ and $U(n)$ exchange positions. This structural relationship

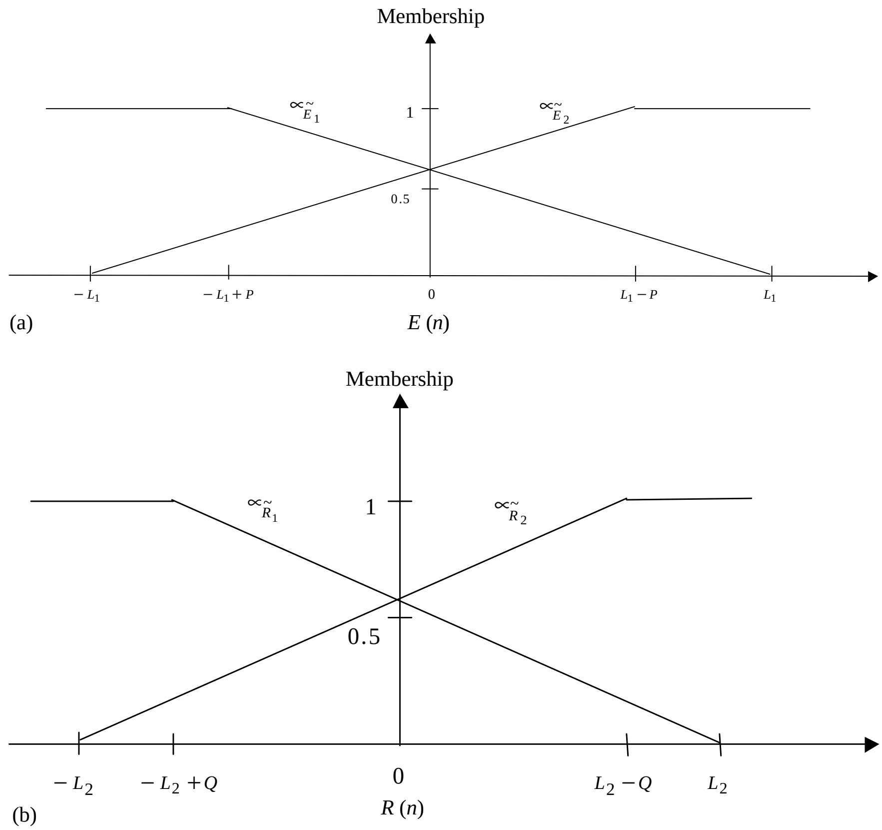

Fig. 6. One specific configuration of the input fuzzy sets shown in Fig. 3.

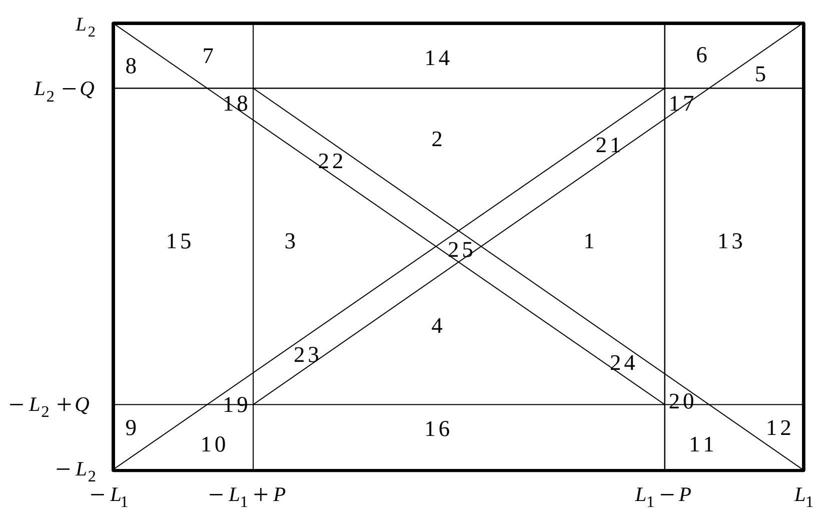

Fig. 7. Superimpose the divisions shown in Figs. 4(a)–(d) with the following modifications: $a_1 = -a_2 = L_1$, $b_1 = -b_2 = L_2$, $L_1 > L_2$, $P_1 = P_2 = P$, $Q_1 = Q_2 = Q$, $P > Q$, and $L_1 - P > L_2 - Q$.

between the PI and PD controllers is important for the structural derivation of the fuzzy PI and PD controllers not only in this paper but also in the literature. Our point is that the structure results developed for fuzzy PI control can directly be extended to the corresponding fuzzy PD control, and vice versa. Consequently, it suffice to study either fuzzy PI control or fuzzy PD control, but not both.

We extend our theorem to include the generalized fuzzy PD controller and its various configurations.

Corollary. *The generalized fuzzy PD controller is a nonlinear PD controller with variable gain and variable control offset.*

Table 3
Result of Zadeh AND operation in each fuzzy rule for the overall input space division shown in Fig. 7

IC No.	Rule 1	Rule 2	Rule 3	Rule 4
1–16	Same as those in Table 1			
17	$\mu_{\tilde{R}_2}$ (A5)	$\mu_{\tilde{R}_1}$ (B3)	$\mu_{\tilde{E}_1}$ (C5)	$\mu_{\tilde{R}_1}$ (D2)
18	$\mu_{\tilde{E}_2}$ (A3)	$\mu_{\tilde{R}_1}$ (B2)	$\mu_{\tilde{R}_2}$ (C3)	$\mu_{\tilde{R}_1}$ (D1)
19	$\mu_{\tilde{R}_2}$ (A4)	$\mu_{\tilde{E}_2}$ (B1)	$\mu_{\tilde{R}_2}$ (C3)	$\mu_{\tilde{R}_1}$ (D1)
20	$\mu_{\tilde{R}_2}$ (A5)	$\mu_{\tilde{R}_1}$ (B3)	$\mu_{\tilde{R}_2}$ (C4)	$\mu_{\tilde{E}_1}$ (D3)
21	$\mu_{\tilde{R}_2}$ (A4)	$\mu_{\tilde{R}_1}$ (B2)	$\mu_{\tilde{E}_1}$ (C5)	$\mu_{\tilde{R}_1}$ (D2)
22	$\mu_{\tilde{E}_2}$ (A3)	$\mu_{\tilde{R}_1}$ (B2)	$\mu_{\tilde{R}_2}$ (C4)	$\mu_{\tilde{R}_1}$ (D2)
23	$\mu_{\tilde{R}_2}$ (A4)	$\mu_{\tilde{E}_2}$ (B1)	$\mu_{\tilde{R}_2}$ (C4)	$\mu_{\tilde{R}_1}$ (D2)
24	$\mu_{\tilde{R}_2}$ (A4)	$\mu_{\tilde{R}_1}$ (B2)	$\mu_{\tilde{R}_2}$ (C4)	$\mu_{\tilde{E}_1}$ (D3)
25	$\mu_{\tilde{R}_2}$ (A4)	$\mu_{\tilde{R}_1}$ (B2)	$\mu_{\tilde{R}_2}$ (C4)	$\mu_{\tilde{R}_1}$ (D2)

Table 4
Analytical structure of the generalized fuzzy PI controller for the overall input space division shown in Fig. 7

IC No.	$\Delta U(n)=$
1 & 13 (use +), 3 & 15 (use −)	$K_{\Delta u}\dfrac{(H_1+H_2)(2L_2-Q)K_e e(n)+(H_1-H_2)(2L_1-P)K_r r(n)\pm(H_1+H_2)(L_1Q-L_2P)}{8L_1L_2-2L_1Q-2L_2P-(4L_2-2Q)K_e\lvert e(n)\rvert}$
2 & 14 (use −), 4 & 16 (use +)	$K_{\Delta u}\dfrac{(H_1+H_2)(2L_2-Q)K_e e(n)+(H_1-H_2)(2L_1-P)K_r r(n)\mp(H_1-H_2)(L_1Q-L_2P)}{8L_1L_2-2L_1Q-2L_2P-(4L_1-2P)K_r\lvert r(n)\rvert}$
5 (use +), 9 (use −)	$K_{\Delta u}\dfrac{(H_1+H_2)(2L_2-Q)K_e e(n)-H_2(2L_1-P)K_r r(n)\pm[2H_1L_1L_2-(H_1-H_2)L_1Q-(2H_1+H_2)L_2P+H_1PQ]}{10L_1L_2-4L_1Q-3L_2P+PQ-(4L_2-2Q)K_e\lvert e(n)\rvert-(2L_1-P)K_r\lvert r(n)\rvert}$
6 (use +), 10 (use −)	$K_{\Delta u}\dfrac{H_2(2L_2-Q)K_e e(n)+(H_1-H_2)(2L_1-P)K_r r(n)\pm[2H_1L_1L_2-(2H_1-H_2)L_1Q-(H_1+H_2)L_2P+H_1PQ]}{10L_1L_2-3L_1Q-4L_2P+PQ-(2L_2-Q)K_e\lvert e(n)\rvert-2(2L_1-P)K_r\lvert r(n)\rvert}$
7 (use −), 11 (use +)	$K_{\Delta u}\dfrac{H_1(2L_2-Q)K_e e(n)+(H_1-H_2)(2L_1-P)K_r r(n)\mp[2H_2L_1L_2-(H_1-2H_2)L_1Q-(H_1+H_2)L_2P+H_2PQ]}{10L_1L_2-3L_1Q-4L_2P+PQ-(2L_2-Q)K_e\lvert e(n)\rvert-2(2L_1-P)K_r\lvert r(n)\rvert}$
8 (use −), 12 (use +)	$K_{\Delta u}\dfrac{(H_1+H_2)(2L_2-Q)K_e e(n)+H_1(2L_1-P)K_r r(n)\mp[2H_2L_1L_2-(H_1-H_2)L_1Q-(H_1+2H_2)L_2P+H_2PQ]}{10L_1L_2-4L_1Q-3L_2P+PQ-2(2L_2-Q)K_e\lvert e(n)\rvert-(2L_1-P)K_r\lvert r(n)\rvert}$
17 & 21 (use +), 19 & 23 (use −)	$K_{\Delta u}\dfrac{H_2(2L_2-Q)K_e e(n)+(2H_1-H_2)(2L_1-P)K_r r(n)\pm H_2(L_1Q-L_2P)}{8L_1L_2-L_1Q-3L_2P-(2L_2-Q)K_e\lvert e(n)\rvert-(2L_1-P)K_r\lvert r(n)\rvert}$
18 & 22 (use −), 20 & 24 (use +)	$K_{\Delta u}\dfrac{H_1(2L_2-Q)K_e e(n)+(H_1-2H_2)(2L_1-P)K_r r(n)\mp H_1(L_1Q-L_2P)}{8L_1L_2-L_1Q-3L_2P-(2L_2-Q)K_e\lvert e(n)\rvert-(2L_1-P)K_r\lvert r(n)\rvert}$
25	$\dfrac{(H_1-H_2)K_r r(n)}{2L_2}$

4. Applicability of the general technique to other fuzzy controllers

4.1. Fuzzy controllers using more than two trapezoidal fuzzy sets for each input variable

The structure derivation technique developed in this paper is not limited to the fuzzy controllers using only two fuzzy sets for each input variable. It is applicable to much more complicated fuzzy controllers, Mamdani type or TS type, that use at least three arbitrary trapezoidal fuzzy sets for each input variable. For example, it can be employed to derive the analytical structure of a fuzzy controller using the fuzzy sets shown in Fig. 2. This is because at any sampling instance, only two fuzzy sets are involved in the fuzzification of an input variable. These two fuzzy sets are neighboring each other. Thus, one can derive the analytical structure by considering only two neighboring input fuzzy sets for each input variable a time. Doing this is the same as we have done with the two fuzzy sets shown in Fig. 3. Specifically, suppose that there are p and q fuzzy sets for $E(n)$ and $R(n)$, respectively, where p and q are greater than two. Then, one needs to consider the ith and $(i+1)$th fuzzy sets for $E(n)$ (see the two fuzzy sets in $[a_1, a_2]$ in Fig. 2) and the jth and $(j+1)$th fuzzy sets for $R(n)$ a time, where $i=1,\ldots,p-1$ and $j=1,\ldots,q-1$. Also, only four fuzzy rules will be involved, the same amount as the generalized fuzzy PI controller above uses. Treating the ith and $(i+1)$th fuzzy sets as $\tilde{E}_1$ and $\tilde{E}_2$ in Fig. 3(a) and the jth and $(j+1)$th fuzzy sets as $\tilde{R}_1$ and $\tilde{R}_2$ in Fig. 3(b), one can divide the two-dimensional space covered by the four fuzzy sets into a number of regions for the evaluation of Zadeh fuzzy AND operation. The actual division depends on the fuzzy sets and may be different from those shown in Fig. 5. After deriving the structure for all the combinations of the pairs of the neighboring input fuzzy sets for all input fuzzy sets, one gets the complete structure of the fuzzy controller. For more detailed information, one is referred to our previous work (e.g., Ying, 1993b; Ying, 1999; Ding, Ying, & Shao, 1999; Ying, 1998) in which we used the original structure-deriving technique to establish the analytical structure for various fuzzy controllers of either Mamdani type or TS type that use input fuzzy sets similar to those shown in Figs. 1(a) and (b).

Likewise, the new technique is not restricted to fuzzy controllers with two input variables. However, deriving the analytical structure for a fuzzy controller involving more than two input variables is challenging. If there are N variables, one must divide N-dimensional space into many N-dimensional regions (i.e., ICs) to evaluate Zadeh AND operation. We have shown in this paper how to accomplish such division when N is two (i.e., Fig. 5). When N is three. The three-dimensional input space must be divided into a number of three-dimensional IC regions for each fuzzy rule first (there should be $2^3 = 8$ rules). This step is much like what is shown in Fig. 4 except each IC region is no longer two-dimensional, but three-dimensional. Then, these eight individual space divisions will be superimposed to generate an overall three-dimensional division for all the eight fuzzy rules. The result will be like what Fig. 5 shows except it will be three-dimensional. Obviously, this procedure is applicable to four or more input variables. Although the derivation is tractable in principle, the difficult level increases dramatically with N. This is because: (1) the number of fuzzy rules, 2^N, increase quickly with N, and (2) it is difficult to visualize the division in a space higher than three-dimensional.

4.2. Equal effectiveness of new technique to Mamdani and TS fuzzy controllers

The general technique is applicable to both Mamdani fuzzy controllers and TS fuzzy controllers. As a matter of fact, it is independent of controller type. This is because the two types of controllers only differ in the rule consequent, one uses fuzzy sets while the other uses functions of the input variables. The general technique is needed for Zadeh fuzzy AND operation in the rule antecedent, which is before the rule consequent even gets involved in the structure derivation. Hence, the real challenge lies in getting Zadeh AND operation result, irrespective to controller type.

Once the membership function due to the AND operation is obtained for every fuzzy rule and every region defined by an overall input space division (which can be N-dimensional), the rest of the analytical structure derivation is quite straightforward and rather similar for Mamdani type and TS type of fuzzy controllers. See, for instance, how we obtain the analytical structures in Table 2 from Zadeh AND results in Table 1. This is a case of a Mamdani fuzzy controller. We now also show this point by a more concrete example of a TS fuzzy controller. Suppose that the fuzzy controller defined in Section 4.1 is a TS fuzzy controller with the following rules:

IF $e(n)$ is Positive AND $r(n)$ is Positive
THEN $\Delta u(n) = c_1 e(n) + d_1 r(n) + g_1$.
IF $e(n)$ is Positive AND $r(n)$ is Negative
THEN $\Delta u(n) = c_2 e(n) + d_2 r(n) + g_2$.
IF $e(n)$ is Negative AND $r(n)$ is Positive
THEN $\Delta u(n) = c_3 e(n) + d_3 r(n) + g_3$.
IF $e(n)$ is Negative AND $r(n)$ is Negative
THEN $\Delta u(n) = c_4 e(n) + d_4 r(n) + g_4$.

The analytical structure for this TS fuzzy controller can be easily derived because the results of Zadeh AND operation are already available in Table 1. As an example, the structure for IC2 is

$$\Delta U(n) = \frac{\mu_{\tilde{E}_2}(c_1 e(n) + d_1 r(n) + g_1) + \mu_{\tilde{R}_1}(c_2 e(n) + d_2 r(n) + g_2) + \mu_{\tilde{E}_1}(c_3 e(n) + d_3 r(n) + g_3) + \mu_{\tilde{R}_1}(c_4 e(n) + d_4 r(n) + g_4)}{\mu_{\tilde{E}_2} + \mu_{\tilde{R}_1} + \mu_{\tilde{E}_1} + \mu_{\tilde{R}_1}}.$$

After simplification and some algebraic manipulations, we obtain an input–output relation, which is rather long in this case. We omit the actual expression here for brevity.

We should also point out that the applicability of our new technique does not depend on the output fuzzy sets, fuzzy inference method, and defuzzifier. This is because the issues that the technique resolves (i.e., arbitrary trapezoidal fuzzy sets and Zadeh AND operation) occur before the output fuzzy sets, fuzzy inference, and defuzzification become involved in the computation of the control action. The technique is applicable to fuzzy controllers that use any type of output fuzzy sets, inference method, and defuzzifier.

5. Results in the literature as special cases of the new results

The derivation technique as well as the analytical structures presented above are not only novel but also much more general than what the existing two techniques have produced.

Among all the new results in this paper, the one in Table 4 is the least general. Yet it is still much more general than the structure results in the current literature and contains them as special cases. We now demonstrate this point. In Table 4, letting $L_1 = L_2 = L$, $P = Q = 0$, $H_1 = L$, and $H_2 = 0$, then only IC1 to IC4 will remain. The analytical structure for such a fuzzy PI controller is

$$\Delta U(n) = \begin{cases} \dfrac{0.5LK_{\Delta u}(K_e e(n) + K_r r(n))}{2L - K_e|e(n)|} & \text{IC1 and IC3,} \\[2ex] \dfrac{0.5LK_{\Delta u}(K_e e(n) + K_r r(n))}{2L - K_r|r(n)|} & \text{IC2 and IC4} \end{cases}$$

which is identical to one of the results that we reported in Ying et al. (1990). On the other hand, if we let $L_1 = L_2 = L$, $P = Q = 0$, and $H_2 = 0$ only, the L in both numerators will be H_1. The expression will be the same as one result in Ying (1993a).

Using the relationship between PI and PD controllers stated above, the analytical structure of the fuzzy PD controller can be immediately obtained, which is

$$U(n) = \begin{cases} \dfrac{0.5LK_{\Delta u}(K_e e(n) + K_r r(n))}{2L - K_e|e(n)|} & \text{IC1 and IC3,} \\[2ex] \dfrac{0.5LK_{\Delta u}(K_e e(n) + K_r r(n))}{2L - K_r|r(n)|} & \text{IC2 and IC4,} \end{cases}$$

which is, in spirit, the same as those in the literature (e.g., Malki et al., 1994).

6. Conclusions

Extending our original structure-deriving technique, we have developed a new and more general technique capable of rigorously deriving the analytical structure for any fuzzy controller, Mamdani type or TS type, as long as it employs the trapezoidal input fuzzy sets and Zadeh fuzzy AND operator. We have demonstrated the technique by applying it to different configurations of the generalized fuzzy PI and PD controllers and have showed the derived structures. Regardless of the configurations, the generalized fuzzy PI (or PD) controller is proved to be a nonlinear PI (or PD) controller with variable gain and variable control offset. Even the least general new result is still much more general than the structures reported in the current literature and contains them as special cases.

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