Available online at www.sciencedirect.com
SCIENCE DIRECT®
Automatica 40 (2004) 1551 – 1559
automatica
www.elsevier.com/locate/automatica

Brief paper

Structural analysis of fuzzy controllers with nonlinear input fuzzy sets in relation to nonlinear PID control with variable gains☆

Amin Haj-Ali, Hao Ying*

Department of Electrical and Computer Engineering, Wayne State University, Detroit, MI 48202, USA

Received 30 August 2002; received in revised form 6 May 2003; accepted 30 March 2004

Abstract

The popular linear PID controller is mostly effective for linear or nearly linear control problems. Nonlinear PID controllers, however, are needed in order to satisfactorily control (highly) nonlinear plants, time-varying plants, or plants with significant time delay. This paper extends our previous papers in which we show rigorously that some fuzzy controllers are actually nonlinear PI, PD, and PID controllers with variable gains that can outperform their linear counterparts. In the present paper, we study the analytical structure of an important class of two- and three-dimensional fuzzy controllers. We link the entire class, as opposed to one controller at a time, to nonlinear PI, PD, and PID controllers with variable gains by establishing the conditions for the former to structurally become the latter. Unlike the results in the literature, which are exclusively for the fuzzy controllers using linear fuzzy sets for the input variables, this class of fuzzy controllers employs nonlinear input fuzzy sets of arbitrary types. Our structural results are thus more general and contain the existing ones as special cases. Two concrete examples are provided to illustrate the usefulness of the new results.

Keywords: Fuzzy control; PID control; Variable gain control; Analytical structure; Nonlinear fuzzy sets

1. Introduction

The linear proportional-integral-derivative (PID) controller has been used to control about 90% industrial processes worldwide (Åström & Hägglund, 2001; Deshpande, 1991). It is still an active and fruitful research topic even though the controller was first invented many decades ago (Åström & Hägglund, 1995; Bennett, 2001). While being adequate for linear or nearly linear applications, this controller is known to perform poorly when the plant to be controlled is (highly) nonlinear, time varying, or with large time delay. For better performance, nonlinear PID control must be used. For instance, the nonlinear PID controllers with anti-windup schemes (e.g., conditional integration and limited integrator) can resolve the integrator windup problem due to actuator saturation (Bohn & Atherton, 1995). Various nonlinear PID controllers have been proposed for better handling of the complex plant characteristics on the basis of the conventional control techniques (e.g., Huang & Han, 2002; Jiang & Gao, 2001; Ni, Jiao, Chen, & Zhang, 1998; Parra-Vega & Arimoto, 2001; Santibanez & Kelly, 1998) and neural networks (e.g., Tan, Dang, & van Cauwenberghe, 1999; Chen & Chang, 1996).

Fuzzy controllers (Ying, 2000b; Yager & Filev, 1994) have gained widespread applications worldwide. Numerous applications have proven fuzzy control to be a cost-effective way to solve challenging practical problems with reduced time (Yen, Langari, & Zadeh, 1995). Through rigorous mathematical analysis, we first proved in the last 1980s that (1) some simple Mamdani fuzzy controllers were structurally nonlinear PI or PD controllers with variable gains, and (2) it was the variable gains that enabled the fuzzy controllers to outperform their linear counterparts when controlling nonlinear or time-delay plants (Ying, Siler, & Buckley, 1990) (see Ying, 1993a, 2000b also). (If at least one of the gains varies with input variable(s), the PID controller becomes nonlinear and the gain is called the variable gain.) The theoretically determined advantage of the

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Gary G. Yen under the direction of Editor Robert R. Bitmead.

* Corresponding author. Tel.: +1-313-577-3738; fax: +1-313-577-1101.
E-mail address: hao.ying@wayne.edu (H. Ying).

0005-1098/$ - see front matter © 2004 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2004.03.019

Mamdani fuzzy PI controller was realized and confirmed when it was used for real-time feedback control of mean arterial pressure in open-heart surgery patients at the Cardiac Surgical Intensive Care Unit (Ying, McEachern, & Sheppard, 1992). The fuzzy controller was judged to outperform a nonlinear PI controller that had been well established for the same application through thousands of patient trials. The proportional and integral gains of that PI controller changed with the state of the blood pressure based on a nonlinear gain table developed by the controller designer (Sheppard, 1980).

Additionally, the input–output relationship of some other Mamdani fuzzy controllers has also been revealed and linked to that of the PID control (Chen, Wang, Hsieh, & Chang, 1998; Hajjaji & Rachid, 1994; Lewis & Liu, 1996; Li & Gatland, 1995; Mann, Hu, & Gosine, 1999; Wong, Chou, & Mon, 1993). The controllers in all these investigations, including Ying et al. (1990) and Ying (1993a), have two common aspects. First, they all used linear fuzzy sets for fuzzifying the input variables. Nonlinear fuzzy sets have not been studied except in a very recent paper of ours (Ying, 2003). Second, the input–output relationship had to be *explicitly* derived for each fuzzy controller before one would even know whether the structure connection with the PID control exists or not.

We have also related the Takagi–Sugeno fuzzy controllers that use linear or nonlinear input fuzzy sets to the PID control (Ying, 1998a, b, 2000a). These are the only results available in the literature for the TS fuzzy controllers. The performance advantages over the linear PID control were shown through computer simulation of the above medical control problem (Ying, 2000a).

In addition to the motivation stated above regarding the importance and advantages of nonlinear PID control, another rational for our study was related to the methodological aspects of nonlinear system theory development. It is well known that unlike linear systems theory, there does not, and will not, exist a general theory for nonlinear systems. Nonlinear systems can only be studied class-by-class at best and system-by-system at worst. This constraint is totally applicable to fuzzy controllers as they are nonlinear. That is to say that there will not be a general theory for nonlinear fuzzy controllers. Therefore, it is important to understand how different choices of various components of fuzzy controllers (e.g., input fuzzy sets in this paper) will affect the nonlinear classification of the controllers. Such understanding can lead to more effective analysis and design techniques tailored to each class of fuzzy controllers. Conventional nonlinear control system techniques have been developed in this way. It should be noted that controller classification is simple for conventional systems because their structures are always explicitly known. In contrast, there is little structural understanding of most fuzzy controllers, especially those use nonlinear components (e.g., nonlinear input fuzzy sets). The objective of the present work was to classify the fuzzy controllers, with respect to input fuzzy sets, into two classes: nonlinear PID controllers and non-PID controllers.

This paper, concerning an important class of Mamdani fuzzy controllers, substantially extends the existing results in the literature. It addresses the following essential issues:

1. Can fuzzy controllers with nonlinear input fuzzy sets also structurally become nonlinear PI, PD, or PID controllers with variable gains? Nonlinear fuzzy sets, such as Gaussian functions and logistic functions, are important and widely used. By "structurally become nonlinear PI, PD, or PID controllers," we mean that a controller whose input–output relation can be expressed, *explicitly or implicitly*, in the form of PI, PD, or PID control with the gains being variable changing with input variables. This representation must be genuine in that the relation cannot be further simplified.
2. If the answer is yes, what are the conditions for this to happen? Is it possible to judge such occurrence without derivation of the input–output relation of the fuzzy controller? The relation is determined by input fuzzy sets, output fuzzy sets, fuzzy rules, fuzzy inference, fuzzy logic operators, and defuzzifier. Different configurations result in different relations. The answer to this question is important because the structural derivation is difficult or even impossible for many configurations.

2. Configuration of fuzzy controllers

The input variables involved are error, error rate, and error acceleration

$$e(n) = S(n) - y(n), \quad r(n) = e(n) - e(n-1),$$

$$d(n) = r(n) - r(n-1),$$

where $S(n)$ is the reference output signal of the plant controlled at sampling time n and $y(n)$ is the output of the plant. We will refer to a fuzzy controller using only $e(n)$ and $r(n)$ or $e(n)$ and $d(n)$ as a two-dimensional fuzzy controller. A fuzzy controller utilizing all the three variables is called a three-dimensional fuzzy controller.

The variables are scaled by design parameters K_e, K_r, and K_d. The scaled variables $E(n) = K_e e(n)$, $R(n) = K_r r(n)$, and $D(n) = K_d d(n)$ fall in the same universe of discourse, $[-L, L]$, of the input fuzzy sets, where L is a design parameter. Each scaled variable is fuzzified by two fuzzy sets, namely "P" (stands for positive) and "N" (stands for negative), which are nonlinear, monotonic, and expandable in Taylor series. If a membership function approaches its limit memberships (0 and 1) asymptotically, L will be infinite. We will use μ, η, and ϑ to denote membership functions for $e(n)$, $r(n)$, and $d(n)$, respectively. The superscript and subscript of these symbols denote, respectively, the class and label (i.e., P and N) of the fuzzy sets. We categorize the fuzzy sets into two classes: classes 1 and 2 (see below). For instance, $\mu_P^1(e)$ signifies a class-1 fuzzy set for $e(n)$ whose

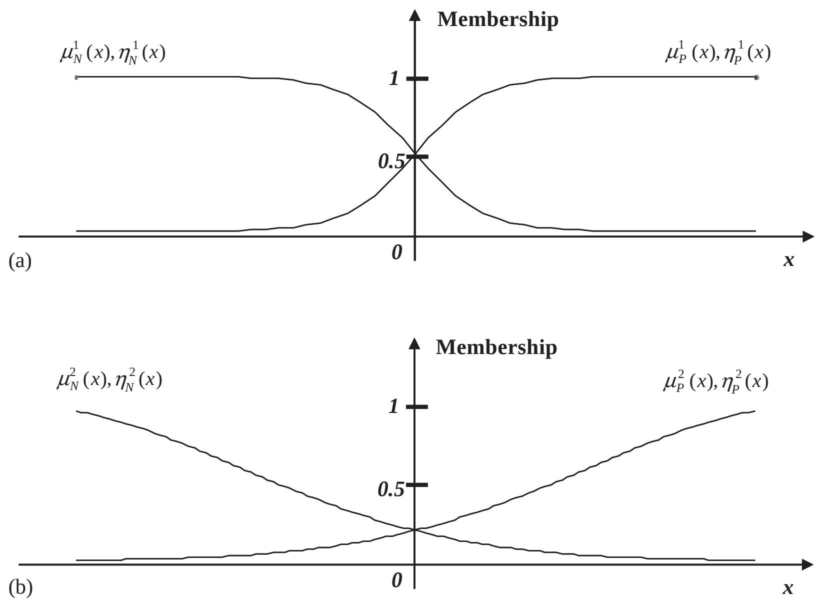


Fig. 1. Classes of input fuzzy sets; (a) examples of class-1 fuzzy sets, (b) examples of class-2 fuzzy sets.

label is P. Whenever the superscript is not present, it means that the fuzzy set can be either class. Also, whenever we use x in replacement of $e(n)$, $r(n)$, or $d(n)$, it means the membership function is for any one of the variables.

A class-1 fuzzy set satisfies the following two conditions (this is also true for the other two input variables):

$$\mu_{\mathrm{P}}^1(e) = 1 - \mu_{\mathrm{N}}^1(e) \quad \text{and} \quad \mu_{\mathrm{P}}^1(0) = \mu_{\mathrm{N}}^1(0) = 0.5. \tag{1}$$

On the other hand, a class-2 fuzzy set satisfies

$$\mu_{\mathrm{P}}^2(e) = \mu_{\mathrm{N}}^2(-e).$$

See Fig. 1 for illustrative examples of the definitions. Obviously, a class-1 set has the property of passing through the point (0, 0.5), whereas a class-2 set can assume any value at the origin of the universe of discourse. Note that the linear fuzzy sets used in the literature (e.g., Ying et al., 1990; Ying, 1993a) are all symmetric and pass through (0, 0.5) and satisfy all the conditions above. Thus, they are special cases and fit the definitions of the both classes.

Fuzzy rules for the two-dimensional controllers are

IF $E(n)$ is P AND $R(n)$ is P THEN $\Delta u(n)$ (or $u(n)$) is P (Rule 1)

IF $E(n)$ is P AND $R(n)$ is N THEN $\Delta u(n)$ (or $u(n)$) is Z (Rule 2)

IF $E(n)$ is N AND $R(n)$ is P THEN $\Delta u(n)$ (or $u(n)$) is Z (Rule 3)

IF $E(n)$ is N AND $R(n)$ is N THEN $\Delta u(n)$ (or $u(n)$) is N (Rule 4)

The output variable of the fuzzy controllers is either $u(n)$ (controller output) or $\Delta u(n)$ (change in controller output). It uses three singleton fuzzy sets, which are labeled as P (for positive), Z (for zero), and N (for negative). Each of them has nonzero membership at only one location of the universes of discourse $(-\infty, \infty)$: H for P, 0 for Z, and $-H$ for N (H is a design parameter).

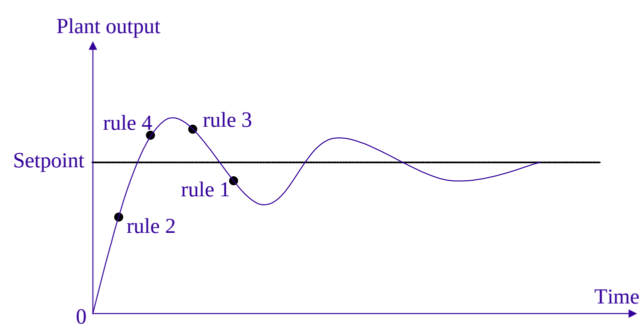


Fig. 2. Illustration of how merely four fuzzy rules can cover all possible situations.

These four rules are sufficient to cover all possible situations, as illustrated in Fig. 2. Rule 1 covers the situation in which plant output is below the setpoint and is still decreasing. Obviously, controller output should be increased. Rule 4 deals with the opposite circumstance and thus controller output should be reduced. There are only two scenarios left: (1) plant output is below the setpoint but is increasing, and (2) plant output is above the setpoint but is decreasing. In either case, it is desirable to let controller output stay at the same level, hoping plant output will land on the setpoint smoothly on its own. This is what Rules 2 and 3 do.

Based on the same reasoning, eight fuzzy rules (Table 1) are enough to cover all the possible situations for the three-dimensional fuzzy controllers. Accordingly, four symmetric pairs of singleton output fuzzy sets are used (Fig. 3). The meanings of the labels are positive large (PL), positive medium (PM), positive small (PS), positive tiny

Table 1
Eight fuzzy rules used by the three-dimensional fuzzy controllers

Rule no.	IF	$E(n)$	AND	$R(n)$	AND	$D(n)$	THEN	$\Delta u(n)$ or $u(n)$
1		P		P		P		PL
2		P		P		N		PM
3		P		N		P		PS
4		P		N		N		PT
5		N		P		P		NT
6		N		P		N		NS
7		N		N		P		NM
8		N		N		N		NL

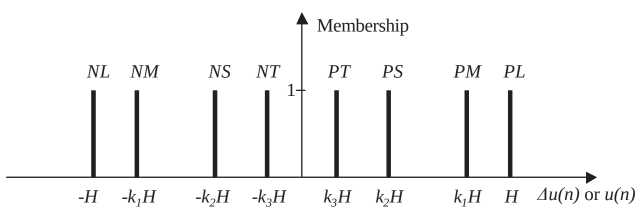


Fig. 3. Singleton output fuzzy sets for the three-dimensional fuzzy controllers.

(PT), negative large (NL), negative medium (NM), negative small (NS), and negative tiny (NT). The parameters k_1, k_2, and k_3 are used to characterize the fuzzy sets and they satisfy $1 \geqslant k_1 \geqslant k_2 \geqslant k_3 \geqslant 0$. The inequalities preserve the symbolic meanings of the fuzzy sets.

The AND operator in all the rules above can be Zadeh-type or the product-type. Although there exist a dozen of AND operators, these two are the only ones that have been used for fuzzy control. As for fuzzy inference method, any type may be used and they will generate the same inference result because of the use of the singleton output fuzzy sets. The popular centroid defuzzifier is used. In the case of $\Delta u(n)$, $u(n) = u(n-1) + \Delta u(n)$. Then, $u(n)$ is fed to the plant.

One may wonder why we do not use more than two fuzzy sets for each input variable. This is because we have already established that when three or more input fuzzy sets are used, a Mamdani fuzzy controller cannot be a (nonlinear) PI, PD, or PID controller (Ying, 1993b). In view of this fact, the class of fuzzy controllers configured above is important and is not restrictive.

3. Conditions for fuzzy controllers to be structurally nonlinear PI, PD, or PID controllers with variable gains

The standard discrete-time PID controller in position form is

$$u(n) = K_\mathrm{p} e(n) + K_\mathrm{i} \sum_{i=0}^{n} e(i) + K_\mathrm{D} r(n)$$

where K_p, K_i, and K_D are proportional gain, integral gain, and derivative gain, respectively. The PID controller is often used in incremental form

$$\Delta u(n) = u(n) - u(n-1) = K_\mathrm{p} r(n) + K_\mathrm{i} e(n) + K_\mathrm{D} d(n).$$

It is straightforward to prove that the PD controller in position form becomes the PI controller in incremental form if (1) $e(n)$ and $r(n)$ exchange positions, (2) K_D is replaced by K_i, and (3) $\Delta u(n)$ and $u(n)$ exchange positions. Consequently, it suffices to study either fuzzy PI control or fuzzy PD control, but not both. We choose to show detailed work on fuzzy PI control.

Below, we express the controller output for the case of class-1 (class-2) fuzzy sets by $\Delta u^1_{\mathrm{PROD}}(n)$ ($\Delta u^2_{\mathrm{PROD}}(n)$) for fuzzy PI control and by $u^1_{\mathrm{PROD}}(n)$ ($u^2_{\mathrm{PROD}}(n)$) for fuzzy PD control. Our focus is on the input space where $E(n)$, $R(n)$, and $D(n)$ are in $[-L, L]$. For a better presentation, the proofs for the lemma, theorems, and corollary are given in the appendix.

3.1. Two-dimensional fuzzy controllers in relation to PI and PD control

We first study the two-dimensional fuzzy controllers that employ product AND operator.

Lemma.

$$\Delta u^1_{\mathrm{PROD}}(n) = (\mu^1_\mathrm{P}(e) + \eta^1_\mathrm{P}(r) - 1)H \quad \text{and}$$

$$u^1_{\mathrm{PROD}}(n) = (\mu^1_\mathrm{P}(e) + \eta^1_\mathrm{P}(r) - 1)H. \tag{2}$$

With the aid of the lemma, we are now ready to establish the condition under which the fuzzy controllers will be a nonlinear PI or PD controller with variable gains.

Theorem 1. *A fuzzy controller that uses $e(n)$ and $r(n)$, the singleton output fuzzy sets, product AND operator, the four fuzzy rules, and the centroid defuzzifier structurally becomes a nonlinear PI or PD controller with variable gains in $[-L, L] \times [-L, L]$ if and only if*

$$\mu^1_\mathrm{P}(x) = 1 - \eta^1_\mathrm{P}(-x) \quad \textit{and} \quad \mu^1_\mathrm{N}(x) = 1 - \eta^1_\mathrm{N}(-x),$$

if the class-1 input fuzzy sets are used, or

$$\mu^2_\mathrm{P}(x) = \eta^2_\mathrm{P}(x) \quad \textit{and} \quad \mu^2_\mathrm{N}(x) = \eta^2_\mathrm{N}(x),$$

if the class-2 input fuzzy sets are employed.

It has been proved in the literature that the fuzzy controllers using linear input fuzzy sets in $[-L, L]$ (as a part of triangular or trapezoidal fuzzy sets) but otherwise are the same as those fuzzy controllers in Theorem 1 are *linear* PI or PD controllers with constant gains (e.g., Ying et al., 1990). Since linear fuzzy sets are special cases of the nonlinear ones, one may wonder whether Theorem 1 contains these previous results as special cases. Indeed, it does, as shown below.

Corollary. *A fuzzy controller that uses class-*1 *fuzzy sets for* $e(n)$ *and* $r(n)$, *the singleton output fuzzy sets, product AND operator, the four fuzzy rules, and the centroid defuzzifier structurally becomes a linear PI or PD controller with constant gains in* $[-L,L]\times[-L,L]$ *if and only if the input fuzzy sets are linear and satisfy Theorem* 1.

We now work on the fuzzy controllers that use Zadeh AND operator in the rules. Unlike the product AND operator, we must first divide $[-L,L]\times[-L,L]$ into a number of input combinations (IC) (Ying et al., 1990). Since the fuzzy sets are nonlinear, the input space dichotomy, which depends on the membership functions, can be complicated and very difficult to achieve in many cases. Thus, we limit our results to sufficient conditions, as opposed to necessary and sufficient ones obtained for the fuzzy controllers using the product AND operator.

Theorem 2. *A fuzzy controller that uses* $e(n)$ *and* $r(n)$, *the singleton output fuzzy sets, Zadeh AND operator, the four fuzzy rules, and the centroid defuzzifier is structurally equivalent to a nonlinear PI or PD controller with variable gains in* $[-L,L]\times[-L,L]$ *if for class-*1 *fuzzy sets,*

$$\mu_{\mathrm{P}}^1(x)=\eta_{\mathrm{P}}^1(x) \quad \text{and} \quad \mu_{\mathrm{N}}^1(x)=\eta_{\mathrm{N}}^1(x)$$

*or for class-*2 *fuzzy sets,*

$$\mu_{\mathrm{P}}^2(x)=\eta_{\mathrm{P}}^2(x) \quad \text{and} \quad \mu_{\mathrm{N}}^2(x)=\eta_{\mathrm{N}}^2(x).$$

3.2. Three-dimensional fuzzy controllers in relation to PID control

The following result relates to Theorem 1, which is about the two-dimensional fuzzy controllers that employ product AND operator.

Theorem 3. *A fuzzy controller that uses* $e(n)$, $r(n)$ *and* $d(n)$ *which satisfies the conditions of Theorem* 1 *pairwisely in terms of* $e(n)$ *and* $r(n)$ *and* $e(n)$ *and* $d(n)$, *the singleton output fuzzy sets, the fuzzy rules in Table* 1, *and the centroid defuzzifier is structurally equivalent to a nonlinear PID controller with variable gains if* $k_3=0$ *and* $k_1+k_2=1$ *in the case of the class-*1 *fuzzy sets, or* $k_1+k_2+k_3=1$ *in the case of the class-*2 *fuzzy sets.*

This result depends on, but is not a generalized version of, Theorem 1, which is for the two-dimensional fuzzy controllers. The conditions established in Theorem 1 are necessary and sufficient, while those set up in Theorem 3 are only sufficient. It seems to be impossible to obtain necessary and sufficient conditions for the three-dimensional fuzzy controllers, even when the product AND operator is used. We do not think that the results for the two-dimensional controllers are generalizable to the three-dimensional controllers. We tried to generalize them, but failed, making us believe the results in this paper not to be generalizable in terms of the number of input variables.

The feasibility of establishing similar conditions for the three-dimensional fuzzy controllers using Zadeh AND operator remains unknown. The three-dimensional cases are significantly more difficult than the two-dimensional ones, regardless which type of AND operator is used. Zadeh AND operator further compounds the difficulty, mainly because of the necessity of the three-dimensional input space dichotomy. The dichotomy depends on the input fuzzy sets. Even if only the classes 1 and 2 fuzzy sets are considered, there are still infinitely many fuzzy sets, resulting in a huge amount of different dichotomies. One must study one by one before the conditions can be established (see the proof of Theorem 2 as an example).

In light of this discussion, we conclude that our results provide as complete analysis as possible for fuzzy PI, PD, and PID controllers that use nonlinear input fuzzy sets.

4. Illustrative examples

Example 1. Consider a two-dimensional fuzzy controller that uses input variables $e(n)$ and $r(n)$, the four fuzzy rules, Zadeh AND operator, and the centroid defuzzifier. The fuzzy sets for $e(n)$ and $r(n)$ are as follows:

$$\mu_{\mathrm{P}}(e)=\frac{1}{1+\exp(-K_e e)}, \quad \mu_{\mathrm{N}}(e)=\frac{1}{1+\exp(K_e e)},$$

$$\eta_{\mathrm{P}}(r)=\frac{\tanh(K_r r)+1}{2} \quad \text{and} \quad \eta_{\mathrm{N}}(r)=\frac{\tanh(-K_r r)+1}{2}$$

$\forall e\in(-\infty,\infty)$ and $\forall r\in(-\infty,\infty)$. $\tanh(x)$ is the hyperbolic tangent function.

Solution. These two fuzzy sets satisfy class-1 criteria. Because the overall functions do not meet the conditions of Theorem 1, we can conclude, without any derivation of the analytical input–output structure of the fuzzy controller, that this controller is not structurally equivalent to a nonlinear PI or PD controller. In other words, the input–output relation of this fuzzy controller cannot be expressed in the form of nonlinear PI or PD controller.

Example 2. Consider a three-dimensional fuzzy controller that employs input variables $e(n)$, $r(n)$, and $d(n)$, product AND operator, the centroid defuzzifier, and the eight fuzzy rules in Table 1. The input fuzzy sets are as follows:

$$\mu_{\mathrm{P}}(e)=\frac{1}{1+\exp(-K_e e)}, \quad \mu_{\mathrm{N}}(e)=\frac{1}{1+\exp(K_e e)},$$

$$\eta_{\mathrm{P}}(r)=\frac{1}{1+\exp(-K_r r)}, \quad \eta_{\mathrm{N}}(r)=\frac{1}{1+\exp(K_r r)},$$

$$\vartheta_{\mathrm{P}}(d)=\frac{1}{1+\exp(-K_d d)} \quad \text{and} \quad \vartheta_{\mathrm{N}}(d)=\frac{1}{1+\exp(K_d d)},$$

which are class-1 fuzzy sets.

Solution. This controller meets the conditions set by Theorem 3. Hence, this fuzzy controller is structurally equivalent to a nonlinear PID controller.

5. Conclusions

We have produced solutions to the two important issues raised in the Introduction. We have developed a new approach for the structure investigation because the existing methods used for the linear input fuzzy sets could not be extended to cover nonlinear input fuzzy sets. The approach makes it possible to study the entire class of the fuzzy controllers as a whole, instead of one-by-one. We have established conditions, some are necessary and sufficient while others are sufficient, under which this class of fuzzy controllers are structurally equivalent to the nonlinear PI, PD, or PID controllers with variable gains. The conditions are easy to be used. No derivation of the input–output relation is needed for any individual fuzzy controller to be tested against the conditions as they cover the entire class. Also, these conditions generalize the existing results in the literature.

In control systems theory, analysis and design are closely related and system stability is always important. Presently, there is no rigorous method in the literature that can design stable fuzzy control systems, locally or globally that involve the fuzzy controllers configured in Section 2. The current practice heavily relies on the trial-and-error and extensive computer simulation. Based on the Lyapunov linearization method, it is straightforward to prove that a fuzzy control system involving a fuzzy controller satisfying Theorem 1, 2, or 3 is locally stable if and only if the linearized system is stable. This local stability result can be practically used for designing fuzzy control systems that are at least locally stable (detail is omitted due to space constraint; see (Ying, 1994, for the principle). This added benefit reflects the necessity and importance of the structure classification of fuzzy controllers in general as well as the structure analysis of the fuzzy controllers in relation to that of the nonlinear PI, PD, and PID control in particular. Together, our new results offer necessary analysis and design tools for realizing, through fuzzy control methodology, more nonlinear PI, PD, and PID controllers to regulate complex plants in practice that are characterized by significant nonlinearity, time variance, and time delay.

Appendix A

Proof of Lemma. Note that for PI control

$$\Delta u^1_{\mathrm{PROD}}(n) = \frac{\mu^1_{\mathrm{P}}(e)\eta^1_{\mathrm{P}}(r)H + \mu^1_{\mathrm{N}}(e)\eta^1_{\mathrm{N}}(r)(-H)}{\mu^1_{\mathrm{P}}(e)\eta^1_{\mathrm{P}}(r) + \mu^1_{\mathrm{P}}(e)\eta^1_{\mathrm{N}}(r) + \mu^1_{\mathrm{N}}(e)\eta^1_{\mathrm{P}}(r) + \mu^1_{\mathrm{N}}(e)\eta^1_{\mathrm{N}}(r)} = \frac{\mathrm{Num}}{\mathrm{Den}}$$

and for PD control

$$u^1_{\mathrm{PROD}}(n) = \frac{\mu^1_{\mathrm{P}}(e)\eta^1_{\mathrm{P}}(r)H + \mu^1_{\mathrm{N}}(e)\eta^1_{\mathrm{N}}(r)(-H)}{\mu^1_{\mathrm{P}}(e)\eta^1_{\mathrm{P}}(r) + \mu^1_{\mathrm{P}}(e)\eta^1_{\mathrm{N}}(r) + \mu^1_{\mathrm{N}}(e)\eta^1_{\mathrm{P}}(r) + \mu^1_{\mathrm{N}}(e)\eta^1_{\mathrm{N}}(r)} = \frac{\mathrm{Num}}{\mathrm{Den}}$$

Using the identity of class-1 fuzzy sets in (1) in the above numerator and denominator,

$$\begin{aligned}\mathrm{Num} &= \mu^1_{\mathrm{P}}(e)\eta^1_{\mathrm{P}}(r)H + \mu^1_{\mathrm{N}}(e)\eta^1_{\mathrm{N}}(r)(-H)\\ &= \mu^1_{\mathrm{P}}(e)\eta^1_{\mathrm{P}}(r)H + (1-\mu^1_{\mathrm{P}}(e))(1-\eta^1_{\mathrm{P}}(r))(-H)\\ &= \mu^1_{\mathrm{P}}(e)\eta^1_{\mathrm{P}}(r)H - \mu^1_{\mathrm{P}}(e)\eta^1_{\mathrm{P}}(r)H + (\mu^1_{\mathrm{P}}(e)+\eta^1_{\mathrm{P}}(r))H - H\\ &= (\mu^1_{\mathrm{P}}(e)+\eta^1_{\mathrm{P}}(r)-1)H,\end{aligned}$$

$$\begin{aligned}\mathrm{Den} &= \mu^1_{\mathrm{P}}(e)\eta^1_{\mathrm{P}}(r) + \mu^1_{\mathrm{P}}(e)\eta^1_{\mathrm{N}}(r)\\ &\quad + \mu^1_{\mathrm{N}}(e)\eta^1_{\mathrm{P}}(r) + \mu^1_{\mathrm{N}}(e)\eta^1_{\mathrm{N}}(r)\\ &= \mu^1_{\mathrm{P}}(e)\eta^1_{\mathrm{P}}(r) + \mu^1_{\mathrm{P}}(e)(1-\eta^1_{\mathrm{P}}(r))\\ &\quad + (1-\mu^1_{\mathrm{P}}(e))\eta^1_{\mathrm{P}}(r) + (1-\mu^1_{\mathrm{P}}(e))(1-\eta^1_{\mathrm{P}}(r))\\ &= 1. \qquad \square\end{aligned}$$

Proof of Theorem 1. Due to the space constraint, we will only prove for the class-1 input fuzzy sets. The class-2 fuzzy set case can be proved similarly. Expand the fuzzy sets in Taylor series:

$$\mu_{\mathrm{P}}(e) = a_{10} + a_{11}K_e e(n) + a_{12}(K_e e(n))^2 + a_{13}(K_e e(n))^3 + \cdots,$$

$$\eta_{\mathrm{P}}(r) = a_{20} + a_{21}K_r r(n) + a_{22}(K_r r(n))^2 + a_{23}(K_r r(n))^3 + \cdots.$$

Applying them to (2) produces (we only prove here the case for $\Delta u^1_{\mathrm{PROD}}(n)$, the proof for $u^1_{\mathrm{PROD}}(n)$ is the same and thus is omitted)

$$\begin{aligned}\Delta u^1_{\mathrm{PROD}}(n) &= (-1 + a_{10} + a_{20} + a_{11}K_e e(n) + a_{21}K_r r(n)\\ &\quad + a_{12}(K_e e(n))^2 + a_{22}(K_r r(n))^2 + a_{13}(K_e e(n))^3\\ &\quad + a_{23}(K_r r(n))^3 + \cdots)\cdot H. \qquad \text{(A.1)}\end{aligned}$$

Note that $\Delta u^1_{\mathrm{PROD}}(n)=0, \forall (K_e e(n)+K_r r(n))=0$. Therefore, replacing $K_r r(n)$ by $-K_e e(n)$, (A.1) will be transformed into a polynomial in $K_e e(n)$

$$\begin{aligned}\Delta u^1_{\mathrm{PROD}}(n) &= (-1 + a_{10} + a_{20} + (a_{11}-a_{21})K_e e(n)\\ &\quad + (a_{12}+a_{22})(K_e e(n))^2 + (a_{13}-a_{23})\\ &\quad \times (K_e e(n))^3 + \cdots)\cdot H\\ &\equiv 0.\end{aligned}$$

Table 2
The result of applying Zadeh fuzzy AND operation to the four fuzzy rules in different ICs

	IC1	IC2	IC3	IC4	IC5	IC6	IC7	IC8	IC9	IC10	IC11	IC12
Rule 1	$\eta_P^1(r)$	$\mu_P^1(e)$	$\mu_P^1(e)$	$\eta_P^1(r)$	$\eta_P^1(r)$	1	$\mu_P^1(e)$	0	0	0	0	0
Rule 2	$\eta_N^1(r)$	$\eta_N^1(r)$	$\mu_P^1(e)$	$\mu_P^1(e)$	$\eta_N^1(r)$	0	0	0	0	0	$\mu_P^1(e)$	1
Rule 3	$\mu_N^1(e)$	$\mu_N^1(e)$	$\eta_P^1(r)$	$\eta_P^1(r)$	0	0	$\mu_N^1(e)$	1	$\eta_P^1(r)$	0	0	0
Rule 4	$\mu_N^1(e)$	$\eta_N^1(r)$	$\eta_N^1(r)$	$\mu_N^1(e)$	0	0	0	0	$\eta_N^1(r)$	1	$\mu_N^1(e)$	0

To realize this equality, the coefficients of all the $K_e e(n)$ terms must be zero simultaneously, leading to $a_{10} + a_{20} = 1$ and

$$a_{1(2k-1)} = a_{2(2k-1)} \stackrel{\Delta}{=} a_{2k-1}, \quad a_{1(2k)} = -a_{2(2k)} \stackrel{\Delta}{=} a_{2k} \qquad \text{for } k = 1, 2, 3, \cdots .$$

Here, the sign $\stackrel{\Delta}{=}$ means "defined as." The above conditions mean

$$\mu_P^1(e) = 1 - \eta_P^1(-r) \quad \text{and} \quad \mu_N^1(e) = 1 - \eta_N^1(-r).$$

Replacing all these results back in (A.1),

$$\Delta u_{PROD}^1(n) = (a_1(K_e e(n) + K_r r(n)) + a_2((K_e e(n))^2 - (K_r r(n))^2) + a_3((K_e e(n))^3 + (K_r r(n))^3) + \cdots) \cdot H.$$

The proof is then completed by a direct application of the binomial rules to this equation, which allows us to factor out $K_e e(n) + K_r r(n)$ in the following equation:

$$\Delta u_{PROD}^1(n) = \beta_{PROD}^1(e, r)(K_e e(n) + K_r r(n)), \tag{A.2}$$

where

$$\beta_{PROD}^1(e, r) = H(a_1 + a_2(K_e e(n) - K_r r(n)) + a_3((K_e e(n))^2 - K_e e(n) K_r r(n) + (K_r r(n))^2) + \theta_{PROD}^1), \tag{A.3}$$

θ_{PROD}^1 represents the error when truncating the series. The variable proportional gain and integral gain are $\beta_{PROD}^1(e, r)K_r$ and $\beta_{PROD}^1(e, r)K_e$, respectively. □

Proof of Corollary. Examining Eq. (A.3), the variable gains become constant gains if and only if $a_i = 0,\ \forall i \geqslant 2$, which in turn implies that the input fuzzy sets must be linear. The control action would then be written as

$$\Delta u_{PROD}^1(n) = a_1 H(K_e e(n) + K_r r(n))$$

or

$$u_{PROD}^1(n) = a_1 H(K_e e(n) + K_r r(n)). \qquad \square$$

Proof of Theorem 2. Only the proof when the class-1 input fuzzy sets are used is provided because of the space limit. Fig. 4 shows the different ICs when the two conditions in Theorem 2 are satisfied. In the case where the membership

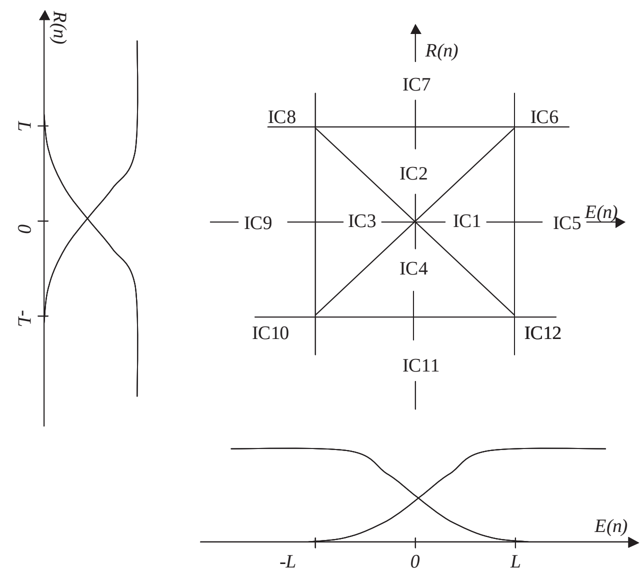


Fig. 4. ICs when using Zadeh AND operator. Input–output structure for each IC must be derived separately.

Table 3
$\Delta u_{ZAD}^1(n)$ in different ICs where $\Delta u_{PROD}^1(n)$ is given in (4)

IC1	$\dfrac{\Delta u_{PROD}^1(n)}{1 + 2\mu_N^1(e)}$	IC5	$\eta_P^1(r)H$	IC9	$-\eta_N^1(r)H$
IC2	$\dfrac{\Delta u_{PROD}^1(n)}{1 + 2\eta_N^1(r)}$	IC6	H	IC10	$-H$
IC3	$\dfrac{\Delta u_{PROD}^1(n)}{1 + 2\eta_P^1(e)}$	IC7	$\mu_P^1(e)H$	IC11	$-\mu_N^1(e)H$
IC4	$\dfrac{\Delta u_{PROD}^1(n)}{1 + 2\eta_P^1(r)}$	IC8	0	IC12	0

functions will approach their limits (0 and 1) only asymptotically as $|L| \to \infty$, a less number of ICs will exist than what the figure shows (i.e., IC5 through IC12 will no longer exist). Nevertheless, we still include here the results in all ICs for completeness. Table 2 shows the result of applying the Zadeh AND operator to Rules 1–4. Putting these AND results into the centroid defuzzifier, the control action in the different ICs is obtained (Table 3). It is evident from Table 3 that the structure of the fuzzy controller in IC1 to IC4 is a nonlinear PI or PD controller with variable gains (just treat the denominators as a part of the variable gains). □

Proof of Theorem 3. We will give the proof for the case when the class-2 fuzzy sets are used. In this case, the denominator of the defuzzifier is not equal to 1. For better presentation, we use the following short notation to the denominator:

$$\begin{aligned}\Xi^2_{\mathrm{PROD}}(e,r,d) &= \mu^2_{\mathrm{P}}(e)\eta^2_{\mathrm{P}}(r)\vartheta^2_{\mathrm{P}}(d) + \mu^2_{\mathrm{P}}(e)\eta^2_{\mathrm{P}}(r)\vartheta^2_{\mathrm{N}}(d)\\ &+ \mu^2_{\mathrm{P}}(e)\eta^2_{\mathrm{N}}(r)\vartheta^2_{\mathrm{P}}(d) + \mu^2_{\mathrm{P}}(e)\eta^2_{N}(r)\vartheta^2_{\mathrm{N}}(d)\\ &+ \mu^2_{\mathrm{N}}(e)\eta^2_{\mathrm{P}}(r)\vartheta^2_{\mathrm{P}}(d) + \mu^2_{\mathrm{N}}(e)\eta^2_{\mathrm{P}}(r)\vartheta^2_{\mathrm{N}}(d)\\ &+ \mu^2_{\mathrm{N}}(e)\eta^2_{\mathrm{N}}(r)\vartheta^2_{\mathrm{P}}(d) + \mu^2_{\mathrm{N}}(e)\eta^2_{\mathrm{N}}(r)\vartheta^2_{\mathrm{N}}(d)\end{aligned}$$

Thus, the control action after defuzzification is (the expression is the same for $u^2_{\mathrm{PROD}}(n)$)

$$\begin{aligned}\Delta u^2_{\mathrm{PROD}}(n) = \frac{H}{\Xi^2_{\mathrm{PROD}}(e,r,d)}\,&(\mu^2_{\mathrm{P}}(e)\eta^2_{\mathrm{P}}(r)\vartheta^2_{\mathrm{P}}(d)\\ &+ k_1\mu^2_{\mathrm{P}}(e)\eta^2_{\mathrm{P}}(r)\vartheta^2_{\mathrm{P}}(-d)\\ &+ k_2\mu^2_{\mathrm{P}}(e)\eta^2_{\mathrm{P}}(-r)\vartheta^2_{\mathrm{P}}(d)\\ &+ k_3\mu^2_{\mathrm{P}}(e)\eta^2_{\mathrm{P}}(-r)\vartheta^2_{\mathrm{P}}(-d)\\ &- k_3\mu^2_{\mathrm{P}}(-e)\eta^2_{\mathrm{P}}(r)\vartheta^2_{\mathrm{P}}(d)\\ &- k_2\mu^2_{\mathrm{P}}(-e)\eta^2_{\mathrm{P}}(r)\vartheta^2_{P}(-d)\\ &- k_1\mu^2_{\mathrm{P}}(-e)\eta^2_{\mathrm{P}}(-r)\vartheta^2_{\mathrm{P}}(d)\\ &- \mu^2_{\mathrm{P}}(-e)\eta^2_{\mathrm{P}}(-r)\vartheta^2_{\mathrm{P}}(-d)).\end{aligned}$$

Putting $k_1 + k_2 + k_3 = 1$ in this equation, using Theorem 1, and arranging the resulting expression, we obtain a nonlinear PID controller with variable gains

$$\begin{aligned}u^2_{\mathrm{PROD}}(n) &= \beta^{2,e}_{\mathrm{PROD}}(e,r,d)K_e e(n) + \beta^{2,r}_{\mathrm{PROD}}(e,r,d)K_r r(n)\\ &+ \beta^{2,d}_{\mathrm{PROD}}(e,r,d)K_d d(n),\end{aligned}$$

$$\begin{aligned}\Delta u^2_{\mathrm{PROD}}(n) &= \beta^{2,e}_{\mathrm{PROD}}(e,r,d)K_e e(n) + \beta^{2,r}_{\mathrm{PROD}}(e,r,d)K_r r(n)\\ &+ \beta^{2,d}_{\mathrm{PROD}}(e,r,d)K_d d(n),\end{aligned}$$

where

$$\begin{aligned}\beta^{2,e}_{\mathrm{PROD}}(e,r,d) &= \frac{k_1\Psi^2_{\mathrm{PROD}}(e,r)}{\Xi^2_{\mathrm{PROD}}(e,r,d)}(\vartheta^2_{\mathrm{P}}(d) + \vartheta^2_{\mathrm{P}}(-d))\\ &\times \beta^2_{\mathrm{PROD}}(e,r) + \frac{k_2\Psi^2_{\mathrm{PROD}}(e,d)}{\Xi^2_{\mathrm{PROD}}(e,r,d)}\\ &\times (\eta^2_{\mathrm{P}}(r) + \eta^2_{\mathrm{P}}(-r))\beta^2_{\mathrm{PROD}}(e,d),\end{aligned}$$

$$\begin{aligned}\beta^{2,r}_{\mathrm{PROD}}(e,r,d) &= \frac{k_1\Psi^2_{\mathrm{PROD}}(e,r)}{\Xi^2_{\mathrm{PROD}}(e,r,d)}(\vartheta^2_{\mathrm{P}}(d) + \vartheta^2_{\mathrm{P}}(-d))\\ &\times \beta^2_{\mathrm{PROD}}(e,r) + \frac{k_3\Psi^2_{\mathrm{PROD}}(r,d)}{\Xi^2_{\mathrm{PROD}}(e,r,d)}\\ &\times (\mu^2_{\mathrm{P}}(e) + \mu^2_{\mathrm{P}}(-e))\beta^2_{\mathrm{PROD}}(r,d),\end{aligned}$$

$$\begin{aligned}\beta^{2,d}_{\mathrm{PROD}}(e,r,d) &= \frac{k_2\Psi^2_{\mathrm{PROD}}(e,d)}{\Xi^2_{\mathrm{PROD}}(e,r,d)}(\eta^2_{\mathrm{P}}(r) + \eta^2_{\mathrm{P}}(-r))\\ &\times \beta^2_{\mathrm{PROD}}(e,d) + \frac{k_3\Psi^2_{\mathrm{PROD}}(r,d)}{\Xi^2_{\mathrm{PROD}}(e,r,d)}\\ &\times (\mu^2_{\mathrm{P}}(e) + \mu^2_{\mathrm{P}}(-e))\beta^2_{\mathrm{PROD}}(r,d).\end{aligned}$$ □

References

Åström, K. J., & Hägglund, T. (1995). *PID controllers: Theory, design and tuning*. Research Triangle Park, N.C.: Instrument Society of America.

Åström, K. J., & Hägglund, T. (2001). The future of PID control. *Control Engineering Practice*, *9*, 1163–1175.

Bennett, S. (2001). The past of PID controllers. *Annual Reviews in Control*, *25*, 43–53.

Bohn, C., & Atherton, D. P. (1995). An analysis package comparing PID anti-windup strategies. *IEEE Control Systems Magazine*, *15*, 34–40.

Chen, C.-L., & Chang, F.-Y. (1996). Design and analysis of neural/fuzzy variable structural PID control systems. *IEE Proceedings—Control Theory and Applications*, *143*, 200–208.

Chen, C.-L., Wang, S.-N., Hsieh, C.-T., & Chang, F.-Y. (1998). Theoretical analysis of crisp-type fuzzy logic controllers using various t-norm sum-gravity inference methods. *IEEE Transactions on Fuzzy Systems*, *6*, 122–136.

Deshpande, P.B. (1991). Improve quality control on-line with PID controllers. *Chemical Engineering Progress*, 71–76.

Hajjaji, A. E., & Rachid, A. (1994). Explicit formulas for fuzzy controller. *Fuzzy Sets and Systems*, *62*, 135–141.

Huang, H., & Han, J.-Q. (2002). Nonlinear PID controller and its applications in power plants. *Proceedings of international conference on power system technology* (pp. 1513–1517).

Jiang, F., & Gao, Z. (2001). An application of nonlinear PID control to a class of truck ABS problems. *Proceedings of the 40th IEEE conference on decision and control* (pp. 516–521).

Lewis, F. L., & Liu, K. (1996). Towards a paradigm for fuzzy logic control. *Automatica*, *32*, 167–181.

Li, H. X., & Gatland, H. B. (1995). A new methodology for designing a fuzzy logic controller. *IEEE Transactions on Systems, Man and Cybernetics*, *25*, 505–512.

Mann, G. K. I., Hu, B.-G., & Gosine, R. G. (1999). Analysis of direct action fuzzy PID controller structures. *IEEE Transactions on Systems, Man and Cybernetics (Part B)*, *29*, 371–388.

Ni, Y., Jiao, L., Chen, S., & Zhang, B. (1998). Application of a nonlinear PID controller on STATCOM with a differential tracker. *Proceedings of international conference on energy management and power delivery* (pp. 29–34).

Parra-Vega, V., & Arimoto, S. (2001). Nonlinear PID control with sliding modes for tracking of robot manipulators. *Proceedings of IEEE international conference on control applications* (pp. 351–356).

Santibanez, V., & Kelly, R. (1998). A class of nonlinear PID global regulators for robot manipulators. *Proceedings of IEEE international conference on robotics and automation* (pp. 3601–3606).

Sheppard, L. C. (1980). Computer control of the infusion of vasoactive drugs. *Annual of Biomedical Engineering*, *8*, 431–444.

Tan, Y.-H., Dang, X.-J., & van Cauwenberghe, A. (1999). Generalised nonlinear PID controller based on neural networks. *Proceedings of information, decision and control* (pp. 519–524).

Wong, C. C., Chou, C. H., & Mon, D. L. (1993). Studies on the output of fuzzy controller with multiple inputs. *Fuzzy Sets and Systems*, *57*, 149–158.

Yager, R. R., & Filev, D. P. (1994). *Essentials of fuzzy modeling and control*. New York: Wiley.

Yen, J., Langari, R., & Zadeh, L. A. (Eds.) (1995). Industrial applications of fuzzy control and intelligent systems. New York: IEEE Press.

Ying, H. (1993a). The simplest fuzzy controllers using different inference methods are different nonlinear proportional-integral controllers with variable gains. *Automatica*, *29*, 1579–1589.

Ying, H. (1993b). General analytical structure of typical fuzzy controllers and their limiting structure theorems. *Automatica*, *29*, 1139–1143.

Ying, H. (1994). Practical design of nonlinear fuzzy controllers with stability analysis for regulating processes with unknown mathematical models. *Automatica*, *30*, 1185–1195.

Ying, H. (1998a). Constructing nonlinear variable gain controllers via the Takagi–Sugeno fuzzy control. *IEEE Transactions on Fuzzy Systems*, *6*, 226–234.

Ying, H. (1998b). The Takagi–Sugeno fuzzy controllers using the simplified linear control rules are nonlinear variable gain controllers. *Automatica*, *34*, 157–167.

Ying, H. (2000a). Theory and application of a novel Takagi–Sugeno fuzzy PID controller. *Information Sciences*, *123*, 281–292.

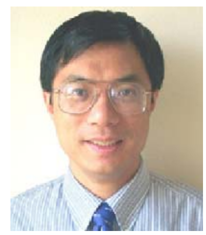
Ying, H. (2000b). *Fuzzy control and modeling: Analytical foundations and applications*. New York: IEEE Press.

Ying, H. (2003). A general technique for deriving analytical structure of fuzzy controllers that use arbitrary trapezoidal/triangular input fuzzy sets and Zadeh fuzzy logic AND operator. *Automatica*, *39*, 1171–1184.

Ying, H., McEachern, M., Eddleman, D., & Sheppard, L. C. (1992). Fuzzy control of mean arterial pressure in postsurgical patients with sodium nitroprusside infusion. *IEEE Transactions on Biomedical Engineering*, *39*, 1060–1070.

Ying, H., Siler, W., & Buckley, J. J. (1990). *Fuzzy control theory: A nonlinear case*, *Automatica*, *26*, 513–520.

Amin A. Haj-Ali received his B.E. degree in Computer and Communications Engineering in 1993 from the American University of Beirut, Beirut, Lebanon, his M.S. degree in Electronics and Computer Control Systems in 1994, and his Ph.D. in Electrical Engineering in 2002 from Wayne State University, Detroit, USA. Since 1996, Dr. Haj-Ali worked in various engineering fields ranging from power generation to broadcast, and IT. Currently, he works at DaimlerChrysler Corporation in the Scientific Labs and Proving Grounds, Auburn Hills, USA.



Dr. Hao Ying is a Professor at the Department of Electrical and Computer Engineering, Wayne State University. He is also a Full Member at Barbara Ann Karmanos Cancer Institute, WSU. Additionally, he serves as an Advisory Professor of Donghua University, Shanghai, China. He was on the faculty of The University of Texas Medical Branch at Galveston between 1992 and 2000. He obtained the B.S. and M.S. degrees in Electrical Engineering from Donghua University in 1982 and 1984, respectively, and the Ph.D. degree in Biomedical Engineering from The University of Alabama at Birmingham in 1990. Besides research in theory and medical applications of fuzzy systems, his current research projects are focused on the development of novel biomedical acoustic sensor systems and medical decision-making techniques, both of which are funded by *National Institutes of Health*.

Professor Ying has published one single-authored research monograph entitled *Fuzzy Control and Modeling: Analytical Foundations and Applications* (318p., IEEE Press, 2000; foreword by Lotfi Zadeh) as well as 62 peer-reviewed journal papers. He is an Associate Editor of the *International Journal of Fuzzy Systems* and a Guest Editor for *Information Sciences* and *Acta Automatica Sinaca*. In 1994, he served as a Program Chair for *The International Joint Conference of North American Fuzzy Information Processing Society (NAFIPS) Conference, Industrial Fuzzy Control and Intelligent System Conference, and NASA Joint Technology Workshop on Neural Networks and Fuzzy Logic*. He has also served as the Publication Chair for the *2000 IEEE International Conference on Fuzzy Systems* and as a Program Committee Member for 17 international conferences. He is the Program Chair for the upcoming *2005 NAFIPS Conference*.