Pergamon

PII: S0005-1098(97)00173-8

Automatica, Vol. 34, No. 2, pp. 157–167, 1998

Printed in Great Britain
0005-1098/98 $19.00 + 0.00

The Takagi–Sugeno Fuzzy Controllers Using the Simplified Linear Control Rules are Nonlinear Variable Gain Controllers

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A simplified linear Takagi–Sugeno fuzzy rule scheme is introduced and the resulting fuzzy controllers are analytically studied in relation to classical controllers, including the PID controller. The properties of the fuzzy controllers are analyzed and graphically demonstrated.

Key Words—Fuzzy control; fuzzy modeling; fuzzy systems; PID controllers; stability; variable structure controllers; variable gain controllers; Takagi–Sugeno.

Abstract—Takagi–Sugeno (TS, for short) fuzzy controllers have been used and treated as black-box controllers, and there exists no explicit structure of any TS fuzzy controller in the literature. In this paper, we investigate the analytical structure of the TS fuzzy controllers that use our newly introduced simplified TS control rule scheme. Our scheme requires that all the rule consequent employ a common linear function and be proportional to one another. The scheme drastically reduces the number of adjustable parameters required by the original TS rule scheme. Product fuzzy logic AND was employed in the rule antecedents. We theoretically proved that the TS controllers using the simplified rule scheme were nonlinear variable gain controllers. The proportionality of the rule consequent parameterized the gain variation characteristics. As special cases, the TS controllers with three or two input variables and the simplified linear rule consequent were nonlinear variable gain PID, PI or PD controllers. As an illustration, we mathematically analyzed the nonlinear variable gain PI controllers, and revealed that the gain variation empowered the controllers to outperform their linear counterparts. Stability of the nonlinear PI control systems was also addressed. A practical example is given to show how to achieve the desired characteristics of the gain variation by properly designing the simplified rules.

1. INTRODUCTION

Fuzzy control (Lee, 1990; Mendel, 1995) has gained significantly wider acceptance in recent years. There exist two major different types of fuzzy control: the Mamdani type (Wang, 1994) and the Takagi–Sugeno (TS, for short) type (Takagi and Sugeno, 1985; Yager and Filev, 1994). Structurally speaking, they differ mainly in the rule consequent of fuzzy control rules. The Mamdani fuzzy controllers utilize fuzzy sets as the consequent while the TS fuzzy controllers employ linear functions as the consequent. Because of the difference, the Mamdani control rules are significantly more linguistically intuitive while the TS rules appear to have more interpolation power even with a relatively small number of control rules. Both types of fuzzy control have successfully been applied to solve practical control problems (Yen *et al.*, 1995).

Compared to the triumphant applications of fuzzy control, fuzzy control theory is far behind. This is in sharp contrast to the well-developed conventional linear and nonlinear control theory that can effectively and efficiently be used to analyze and design control systems with guaranteed stability and control performance required by end users. For various reasons, fuzzy controllers were and are still, to a large extent, utilized and treated as black-box controllers that, when constructed properly by the trial-and-error method, could produce satisfactory control results.

Fuzzy controllers are nonlinear controllers in nature. Thus, to advance fuzzy control theory, establishment of an analytical framework relative to classical control theory is imperative. Within such a framework, fuzzy controllers could explicitly be analyzed and fuzzy control systems could truly be designed in the sense of conventional control. Most importantly, abundant analytical analysis and design tools in nonlinear and linear control theories would readily be used for studying various aspects of fuzzy controllers and fuzzy control systems, such as stability and robustness, just to name a few.

*Received 19 May 1996 received in revised form 12 June 1997. Thin paper was recommended for publication in revised form by Associate Editor K. P. White under the direction of Editor Christine Mitchell. *Corresponding author* Hao Ying. Tel. 1 409 772 8415; Fax 1 409 772 0751; E-mail hying@utmb.edu.

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There has been a growing interest and effort in the analytical study of the Mamdani fuzzy controllers. The research topics range from derivation of the fuzzy controllers' structure relative to classical controllers (e.g. Bouslama and Ichikawa, 1992; Buckley, 1992; Hajjaji and Rachid, 1994; Langari, 1992; Lewis and Liu, 1996; Matia *et al.*, 1992; Pok and Xu, 1993; Wong *et al.*, 1993; Ying *et al.*, 1990; Ying, 1993a, b, 1994a, b) to system stability analysis (e.g. Chen and Ying, 1998; Langari and Tomizuka, 1990; Wang *et al.*, 1990; Ying, 1993a) and to system design (e.g. Malki *et al.*, 1994; Tang and Mulholland, 1987; Ying, 1994a). In comparison, effort in analytical investigation of the TS fuzzy controllers has been rather limited (Tanaka *et al.*, 1996; Wang *et al.*, 1996). There exists no analytical structure of any TS fuzzy controllers in the literature at present, let alone formal relation between the TS fuzzy controllers and classical controllers such as the widely used PID controllers. Generally and philosophically speaking, for any controller including fuzzy controller, the more the structural information is available about the controller, the better and more specific the analysis and design results will be developed, and vice versa.

Compared to the Mamdani fuzzy controllers, analytical study of the TS fuzzy controllers seems to involve significantly more difficulties because, for one thing, the controllers usually have too many adjustable controller parameters. All the parameters in the rule consequent are adjustable and unknown and the number of parameters grows exponentially with the increase of the number of input variables. The parameters, in theory, provide a tuning of local control action, possibly resulting in superior control performance. To a large extent, the power of the TS rule scheme lies in these parameters. However, manual tuning of these parameters in practice could be ineffective, inefficient, inappropriate or sometimes even impossible when the parameters are too many. This is the case partially because of the lack of linguistic intuitiveness of the TS rule consequent. Automatic tuning techniques, such as on-line or off-line parameter optimization schemes, may be ineffective and/or inefficient when the dimension of the parameter space is too high. On-line parameter identification makes the TS fuzzy controllers become adaptive controllers. As with any adaptive controller, control system stability and convergence of controller parameter identification are always central issues of concern. With a large number of unknown parameters, the convergence may be difficult and system stability could be hard to guarantee.

The objectives of our research presented in this paper were threefold. First, we wanted to make the TS control rule scheme simpler and more efficient. We modified the TS rule scheme in such a way that our new scheme, while maintaining the spirit of the TS scheme, drastically reduced the number of parameters in the rule consequent required (see Section 2). Second, we wanted to derive the explicit structure of the TS fuzzy controllers with our newly introduced rule scheme and relate the resulting structures to conventional controllers, including the PID controller and variable gain controllers (see Section 3). Finally, we wanted to study various analytical aspects of the resulting structures in the context of conventional control.

2. CONFIGURATION OF THE TS FUZZY CONTROLLERS IN THIS STUDY

The TS fuzzy controllers under this investigation used M discrete-time input variables ($M \geq 1$): namely $x_1(n), x_2(n), \ldots, x_M(n)$, where n represented sampling time. Each variable was fuzzified by two input fuzzy sets, named "positive" and "negative", respectively. The mathematical definitions of the "positive" and "negative" fuzzy sets were identical for different input variables and were

$$\mu_P(x_i) = \begin{cases} 0, & x_i < -L \\ \dfrac{x_i + L}{2L}, & -L \leq x_i \leq L \\ 1, & x_i > L \end{cases}$$

and (1)

$$\mu_N(x_i) = \begin{cases} 1, & x_i < -L \\ \dfrac{-x_i + L}{2L}, & -L \leq x_i \leq L \\ 0, & x_i > L, \end{cases}$$

where the subscripts P and N represent "positive" and "negative", respectively. Figure 1 illustrates the definitions graphically; and note that the membership value of $\mu_P(x_i)$ and $\mu_N(x_i)$ is either 0 or 1 when x_i is outside the interval $[-L, L]$. The value of L affects the control performance and thus should be carefully chosen during design.

Because each input variable was fuzzified by two input fuzzy sets, there existed 2^M different combinations of the input fuzzy sets. Hence we used 2^M different fuzzy rules to cover all these combinations.

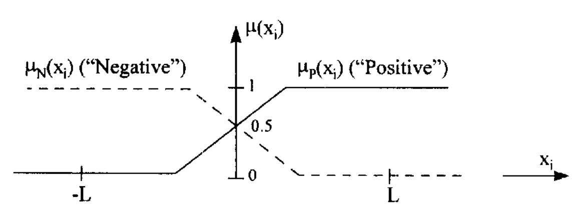

Fig. 1. The graphical definition of the input fuzzy sets used in this paper. All the input variables, denoted as x_i for $i = 1, 2, \ldots, M$, are fuzzified by the identical input fuzzy sets. The corresponding mathematical definitions are given in (1).

In the original TS fuzzy control rules (Takagi and Sugeno, 1985), a rule consequent is always a linear function of input variables [we have already proved that the general TS fuzzy systems with linear rule consequent are universal approximators (Ying, 1998a)]. The jth rule looks like

IF $x_1(n)$ is A_{1j} AND ... AND $x_M(n)$ is A_{Mj} THEN

$$v_j(n) = a_{0j} + a_{1j}x_1(n) + \cdots + a_{Mj}x_M(n) \tag{2}$$

where $v_j(n)$ is a local contribution from the jth rule to the overall controller output and A_{ij}'s are input fuzzy sets. The coefficients a_{ij}'s and a_{0j} are adjustable controller parameters, and can be any value, empowering the rules to generate desired global as well as local control action. However, there is quite a big price to pay for this advantage: there are too many parameters needed to be tuned on line or off line. For 2^M rules, each of which has $M + 1$ parameters, there are in total

$$\kappa = (M + 1) \times 2^M \tag{3}$$

number of parameters. Even when M is small, κ is still a relatively large number. For instance, when $M = 3$, $\kappa = 32$ and when $M = 2$, $\kappa = 12$. Compared to the widely used linear PID, PI and PD controllers which require tuning of only three or two parameters, the TS controllers using rules (2) are extremely at a disadvantage as far as practicality and ease of use are concerned. No human operator could possibly tune 32 or 12 controller parameters effectively and efficiently.

To overcome this disadvantage, we have developed a rule scheme that can significantly reduce the number of parameters in the original TS rules. Specifically, the TS rules in our study were required to be in the following format (R_j means the jth rule where $1 \le j \le 2^M$):

R_1: IF $x_1(n)$ is A_{11} AND ... AND $x_M(n)$ is A_{M1} THEN $v_1(n) = k_1(a_0 + a_1x_1(n) + \cdots + a_Mx_M(n))$

R_2: IF $x_1(n)$ is A_{12} AND ... AND $x_M(n)$ is A_{M2} THEN $v_2(n) = k_2v_1(n)$

.... (4)

R_j: IF $x_1(n)$ is A_{1j} AND ... AND $x_M(n)$ is A_{Mj} THEN $v_j(n) = k_jv_1(n)$.

Practically speaking, the inclusion of k_1 in R_1 is unnecessary because the role of k_1 can always be achieved by letting $a'_0 = k_1a_0$, $a'_1 = k_1a_1, \ldots, a'_M = k_1a_M$, where $a'_0, a'_1, \ldots, a'_M$ would be the new coefficients in R_1 without k_1. For a better notation, however, we used k_1 to make the subscript of k be consistent and run from 1 to 2^M. Without loss of generality, k_1 was always supposed to be one in our study, but we often purposely kept k_1 in our derivation and the results for completeness of the analytical presentation.

For 2^M rules in (4), $a_0, a_1, \ldots, a_M, k_2, k_3, \ldots, k_{2^M}$ are unknown adjustable parameters. Thus, the total number of unknown parameters is

$$\gamma = M + 2^M,$$

which is much smaller than κ in (3) for any given M value. For instance, $\gamma = 11$ (as compared to $\kappa = 32$) when $M = 3$ and $\gamma = 6$ (as compared to $\kappa = 12$) when $M = 2$. We called the TS fuzzy control rules in (4) simplified linear TS fuzzy control rules.

The spirit of our simplified rule scheme is that rule consequent are proportional to one another. The proportionality is fixed for each rule and is k_j for the jth rule. Although the proportionality is fixed for each rule, one should note that all the rule consequent are a linear function of input variables and they always change with input variables. In other words, none of the rule consequent is constant. Compared to the original TS rule scheme, our simplified rule scheme greatly reduces the number of adjustable controller parameters required. As will be shown in the next section, even simplified linear control rules (4) are not restrictive, in the context of control, and they are indeed able to produce superior nonlinear control to what the linear PID, PI, or PD controllers can possibly offer. As a matter of fact, we have proved that the general TS fuzzy systems/controllers with the simplified nonlinear rules are universal approximators (Ying, 1998b).

We used product fuzzy logic AND to blend the M membership values in the rule antecedent of the simplified TS rules in (4) to generate a combined membership for $v_j(n)$, denoted as μ_j:

$$\mu_j = \prod_{i=1}^{M} \mu_\beta(x_i),$$

where the subscript β represented either P ("positive") or N ("negative") in our study.

Finally, we used the popular centroid defuzzifier to combine the local contribution from all the rules, producing an incremental output of the fuzzy controllers, $\Delta u(n)$:

$$\Delta u(n) = \frac{\sum_{j=1}^{2^M} \mu_j v_j(n)}{\sum_{j=1}^{2^M} \mu_j}. \tag{5}$$

Due to the nature of the membership functions of the input fuzzy sets that we employed [i.e. $\mu_P(x_i) + \mu_N(x_i) = 1$], it can easily be proven that the denominator of (5) is always equal to one. Hence, (5) reduces to

$$\Delta u(n) = \sum_{j=1}^{2^M} \mu_j v_j(n).$$

Since 90% industrial processes are currently controlled by the PID, PI or PD controllers (Deshpande, 1991; Ogata, 1992), in our study we focused on the relationship between the TS fuzzy controllers and these popular controllers. A PID controller uses the following three input variables:

$$x_1(n) = \mathrm{SP}(n) - y(n),$$
$$x_2(n) = x_1(n) - x_1(n-1), \quad (6)$$
$$x_3(n) = x_2(n) - x_2(n-1),$$

where SP(n) is a setpoint/reference signal of process output, and $y(n)$ is the process output at sampling time n. $x_1(n)$, $x_2(n)$ and $x_3(n)$ represent the position, velocity and acceleration of the process output at sampling time n. The linear discrete-time PID controller in incremental form is:

$$\Delta u_{\mathrm{PID}}(n) = \bar{K}_{\mathrm{p}}x_2(n) + \bar{K}_{\mathrm{i}}x_1(n) + \bar{K}_{\mathrm{d}}x_3(n), \quad (7)$$

where $\bar{K}_{\mathrm{p}}$, $\bar{K}_{\mathrm{i}}$ and $\bar{K}_{\mathrm{d}}$ are control gains called proportional gain, integral gain and derivative gain, respectively. The PID controller becomes the linear PI controller in incremental form when $\bar{K}_{\mathrm{d}} = 0$:

$$\Delta u_{\mathrm{PI}}(n) = \bar{K}_{\mathrm{p}}x_2(n) + \bar{K}_{\mathrm{i}}x_1(n) \quad (8)$$

or becomes the linear PD controller in incremental form when $\bar{K}_{\mathrm{i}} = 0$:

$$\Delta u_{\mathrm{PD}}(n) = \bar{K}_{\mathrm{p}}x_2(n) + \bar{K}_{\mathrm{d}}x_3(n). \quad (9)$$

One notes that the PI controller becomes the PD controller if $x_1(n)$ is replaced by $x_3(n)$ and $\bar{K}_{\mathrm{i}}$ by $\bar{K}_{\mathrm{d}}$, and vice versa.

For the fuzzy controllers using $x_1(n)$ and $x_2(n)$ in (6) as input variables, assuming $a_0 = 0$, the rules in (4) generated four simplified linear TS control rules as shown in Table 1. The corresponding incremental output of the fuzzy controllers was

$$\Delta u(n) = \sum_{j=1}^{4} \mu_j v_j(n) = v_1(n) \times \sum_{j=1}^{4} \mu_j k_j = F(x_1,x_2)(a_1x_1(n) + a_2x_2(n)) \quad (10)$$

where

$$F(x_1,x_2) = k_1\mu_{\mathrm{P}}(x_1)\mu_{\mathrm{P}}(x_2) + k_2\mu_{\mathrm{P}}(x_1)\mu_{\mathrm{N}}(x_2) + k_3\mu_{\mathrm{N}}(x_1)\mu_{\mathrm{P}}(x_2) + k_4\mu_{\mathrm{N}}(x_1)\mu_{\mathrm{N}}(x_2). \quad (11)$$

Table 1. Four simplified linear TS fuzzy control rules used by the TS fuzzy controllers with $x_1(n)$ and $x_2(n)$ in (6) as input variables

Rule no.	IF $x_1(n)$ is	AND $x_2(n)$ is	THEN
R_1	Positive	Positive	$v_1(n) = k_1(a_1x_1(n) + a_2x_2(n))$
R_2	Positive	Negative	$v_2(n) = k_2v_1(n)$
R_3	Negative	Positive	$v_3(n) = k_3v_1(n)$
R_4	Negative	Negative	$v_4(n) = k_4v_1(n)$

In the rule consequent, $k_1 = 1$, and k_2, k_3 and k_4 are constants.

The fuzzy controllers using $x_1(n)$, $x_2(n)$ and $x_3(n)$ in (6) as input variable had eight simplified linear TS rules (Table 2), generated by (4), a_0 being assumed zero. The corresponding incremental output of the fuzzy controllers was

$$\Delta u(n) = \sum_{j=1}^{8} \mu_j v_j(n) = v_1(n) \times \sum_{j=1}^{8} \mu_j k_j = G(x_1,x_2,x_3)(a_1x_1(n) + a_2x_2(n) + a_3x_3(n)), \quad (12)$$

where

$$\begin{aligned} G(x_1,x_2,x_3) = {} & k_1\mu_{\mathrm{P}}(x_1)\mu_{\mathrm{P}}(x_2)\mu_{\mathrm{P}}(x_3) \\ & + k_2\mu_{\mathrm{P}}(x_1)\mu_{\mathrm{P}}(x_2)\mu_{\mathrm{N}}(x_3) \\ & + k_3\mu_{\mathrm{P}}(x_1)\mu_{\mathrm{N}}(x_2)\mu_{\mathrm{P}}(x_3) \\ & + k_4\mu_{\mathrm{P}}(x_1)\mu_{\mathrm{N}}(x_2)\mu_{\mathrm{N}}(x_3) \\ & + k_5\mu_{\mathrm{N}}(x_1)\mu_{\mathrm{P}}(x_2)\mu_{\mathrm{P}}(x_3) \\ & + k_6\mu_{\mathrm{N}}(x_1)\mu_{\mathrm{P}}(x_2)\mu_{\mathrm{N}}(x_3) \\ & + k_7\mu_{\mathrm{N}}(x_1)\mu_{\mathrm{N}}(x_2)\mu_{\mathrm{P}}(x_3) \\ & + k_8\mu_{\mathrm{N}}(x_1)\mu_{\mathrm{N}}(x_2)\mu_{\mathrm{N}}(x_3). \quad (13) \end{aligned}$$

Generalizing (12) to the TS fuzzy controllers with more than three input variables (i.e. $M > 3$) and 2^M simplified linear TS control rules (again, $a_0 = 0$ was assumed), we obtained the incremental output of the fuzzy controllers as follows:

$$\Delta u(n) = \sum_{j=1}^{2^M} \mu_j v_j(n) = v_1(n) \times \sum_{j=1}^{2^M} \mu_j k_j = H(x_1, \dots, x_M)(a_1x_1(n) + \cdots + a_Mx_M(n)), \quad (14)$$

where $H(x_1, \dots, x_M)$ was a nonlinear function of $x_i(n)$, k_j and L, where $i = 1, \dots, M$ and $j = 1, \dots, 2^M$.

3. ANALYTICAL STRUCTURE OF THE TS FUZZY CONTROLLERS WITH THE SIMPLIFIED CONTROL RULES IN RELATION TO CONVENTIONAL LINEAR AND NONLINEAR CONTROLLERS

3.1. *The TS fuzzy controllers with the simplified control rules are nonlinear variable gain controllers*

Our following results relate the TS fuzzy controllers with the simplified linear TS control rules to nonlinear variable gain controllers, including nonlinear variable gain PID, PI and PD controllers.

Theorem 1. For $x_i \in (-\infty, \infty)$ where $i = 1, 2, \dots, M$, the TS fuzzy controllers using the simplified linear TS control rules, whose output is described by (14), are nonlinear variable gain controllers.

Table 2. Nine simplified linear TS fuzzy control rules used by the TS fuzzy controllers with $x_1(n)$, $x_2(n)$ and $x_3(n)$ in (6) as input variables

Rule no.	IF $x_1(n)$ is	AND $x_2(n)$ is	AND $x_3(n)$ is	THEN
R_1	Positive	Positive	Positive	$v_1(n) = k_1(a_1x_1(n) + a_2x_2(n) + a_3x_3(n))$
R_2	Positive	Positive	Negative	$v_2(n) = k_2v_1(n)$
R_3	Positive	Negative	Positive	$v_3(n) = k_3v_1(n)$
R_4	Positive	Negative	Negative	$v_4(n) = k_4v_1(n)$
R_5	Negative	Positive	Positive	$v_5(n) = k_5v_1(n)$
R_6	Negative	Positive	Negative	$v_6(n) = k_6v_1(n)$
R_7	Negative	Negative	Positive	$v_7(n) = k_7v_1(n)$
R_8	Negative	Negative	Negative	$v_8(n) = k_8v_1(n)$

In the rule consequent, $k_1 = 1$, and k_j's ($j = 2, \ldots, 9$) are constants.

Proof. According to (14), the fuzzy controllers with the simplified linear TS rules are

$$\Delta u(n) = H(x_1, \ldots, x_M)(a_1x_1(n) + \cdots + a_Mx_M(n))$$

which are nonlinear variable gain controllers. The variable gains are $a_iH(x_1, \ldots, x_M)$ for the input variable $x_i(n)$ where $i = 1, 2, \ldots, M$. The gain variation is with respect to the linear controller $a_1x_1(n) + \cdots + a_Mx_M(n)$. □

Corollary 2. As a special case, the fuzzy controllers using three input variables (6) and the simplified linear TS control rules, whose output is described by (12), become nonlinear variable gain PID controllers.

Proof. When the simplified linear TS rules are used and $M = 3$, (14) reduces to (12):

$$\Delta u(n) = G(x_1,x_2,x_3)(a_1x_1(n) + a_2x_2(n) + a_3x_3(n))$$

which, compared to (7), is nonlinear PID controller with variable proportional gain, integral gain and derivative-gain being $a_2G(x_1,x_2,x_3)$, $a_1G(x_1,x_2,x_3)$ and $a_3G(x_1,x_2,x_3)$, respectively. □

The gains vary because the value of $H(x_1, \ldots, x_M)$ (or $G(x_1,x_2,x_3)$) varies. The value is determined by k_j, L and $x_i(n)$. Since k_j and L are constants, the gains vary continuously only with the change of $x_i(n)$ from one sampling time to another. The gain variation characteristics are parameterized by k_j and L and their different values produce different characteristics.

3.2. *The TS fuzzy controllers with the simplified linear TS control rules become linear controllers under some conditions*

We now show that the TS fuzzy controllers with the simplified linear TS control rules become linear controllers, including the linear PID controller, when k_j's take some particular values.

Theorem 3. For $x_i \in (-\infty, \infty)$ where $i = 1, 2, \ldots, M$, the TS fuzzy controllers with the simplified linear TS control rules, whose output is described by (14), become linear controllers if $k_j = \xi$, where ξ is any nonzero number, for $j = 1, 2, \ldots 2^M$.

Proof. When $k_j = \xi$ for all j, the rule consequent of all the 2^M rules become the same: $\xi a_1x_1(n) + \cdots + \xi a_Mx_M(n)$. Note that the antecedents of the rules list all the possible input states. That means, when $k_j = \xi$, $\Delta u(n) = \xi a_1x_1(n) + \cdots + \xi a_Mx_M(n)$, regardless of value of the input variables. That is to say that the fuzzy controllers become linear controllers:

$$\Delta u(n) = \xi a_1x_1(n) + \cdots + \xi a_Mx_M(n). \quad \square$$

Corollary 4. As a special case, the fuzzy controllers using the three input variables in (6) become the linear PID controller.

Proof. When $k_j = \xi$ and $M = 3$, the fuzzy controllers become

$$\Delta u(n) = \xi a_1x_1(n) + \xi a_2x_2(n) + \xi a_3x_3(n).$$

which, according to (7), is the linear PID controller with ξa_2, ξa_1 and ξa_3 being the proportional gain, integral-gain and derivative-gain, respectively. □

Theorem 3 reveals that the TS fuzzy controllers with the simplified linear TS control rules can realize the linear PID, PI and PD controllers as well as the general linear controllers with more than three input variables. Such realization, nevertheless, is merely of theoretical interest with little practical value as one would always be better off in directly designing and implementing a linear controller based on the linear control theory instead of fuzzy control methodology.

In the rest of this section below, we will provide an in-depth analysis of the nonlinear variable gain PI controllers to demonstrate the usefulness of the gain variation in the context of control. There were two main reasons why the nonlinear PI controllers were chosen. First, the PI control is one of the most popular and useful control schemes in industry.

Any controllers that are comparably simple in the gain tuning but have proved potential to significantly outperform it, especially when nonlinear processes are involved, would be of great practical value. Second, gain variation with two input variables is a three-dimensional problem that can be presented and visualized graphically along with analytical analysis. The results obtained may be extended to the more general TS fuzzy controllers. However, if three or more input variables are involved, graphical presentation and visualization of the gain variation with all the input variables is impossible.

3.3. *The TS fuzzy controllers with the simplified linear rules and $x_1(n)$ and $x_2(n)$ as input variables are nonlinear variable gain PI controllers*

Based on (10), the incremental output of the TS fuzzy controllers with $x_1(n)$ and $x_2(n)$ in (6) and the four simplified linear TS control rules given in Table 1 was:

$$\Delta u(n) = F(x_1,x_2)(a_1x_1(n) + a_2x_2(n))$$

which, compared to the linear PI controller (8), was nonlinear PI controllers with

$$K_p(x_1,x_2) = a_2F(x_1,x_2) \quad \text{and} \quad K_i(x_1,x_2) = a_1F(x_1,x_2) \tag{15}$$

being the variable proportional gain and integral-gain, respectively. Similarly, the TS fuzzy controllers with input variables $x_2(n)$ and $x_3(n)$ and the rules in Table 1 with $x_1(n)$ replaced by $x_2(n)$ and $x_2(n)$ by $x_3(n)$ were nonlinear PD controllers (see (9)) with

$$K_p(x_1,x_2) = a_2F(x_1,x_2) \quad \text{and} \quad K_d(x_1,x_2) = a_3F(x_1,x_2)$$

being the variable proportional gain and derivative gain, respectively.

3.3.1. General characteristics of the variable gains of the nonlinear PI controllers. The variable proportional gain and integral gain are essentially determined by the nonlinear function $F(x_1,x_2)$. Thus, in our study we concentrated on characteristics of $F(x_1,x_2)$. The first step was to derive the analytical expression of $F(x_1,x_2)$.

At any sampling time, $x_1(n)$ (and $x_2(n)$) was in one of the following three intervals: $(-\infty, -L)$, $[-L, L]$ and (L, ∞). We divided the $x_1(n)$–$x_2(n)$ plane into nine regions shown in Fig. 2. Substituting the definitions of $\mu_p(x_1)$, $\mu_N(x_1)$, $\mu_p(x_2)$ and $\mu_N(x_2)$ given in (1) into (11) and simplifying the resulting expression, we obtained explicit $F(x_1,x_2)$ for Region 1 in Fig. 2:

$$\begin{aligned} F(x_1,x_2) = \frac{k_1}{4L^2}[&(1 + k_2 + k_3 + k_4)L^2 \\ &+ (1 + k_2 - k_3 - k_4)Lx_1(n) \\ &+ (1 - k_2 + k_3 - k_4)Lx_2(n) \\ &+ (1 - k_2 - k_3 + k_4)x_1(n)x_2(n)]. \end{aligned} \tag{16}$$

This expression along with $F(x_1,x_2)$ corresponding to the rest of the eight regions in Fig. 2 are given in Table 3. $F(x_1,x_2)$ for Region 1 varies with $x_1(n)$ or/and $x_2(n)$. In Regions 2 and 6, $F(x_1,x_2)$ varies only with $x_2(n)$ while in Regions 4 and 8, $F(x_1,x_2)$ changes only with $x_1(n)$. Hence, in Regions 1, 2, 4, 6 and 8, the fuzzy controllers are nonlinear variable gain PI controllers. $F(x_1,x_2)$ is a constant k_1, k_1k_3, k_1k_4 and k_1k_2 in Regions 3, 5, 7 and 9, respectively, making the fuzzy controllers linear PI controllers.

How $F(x_1,x_2)$ varies with the input variables is determined by the values of k_j and L. Their different values produce different types of variable gains and therefore different nonlinear variable gain PI controllers. Among the nine regions, Region 1 is of the most interest and importance because $F(x_1,x_2)$ is most nonlinear in this region. Furthermore, stable

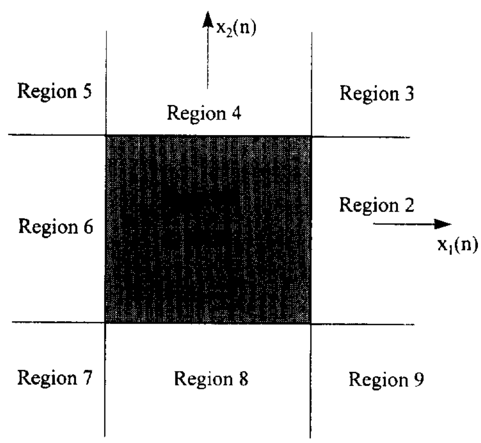

Fig. 2. The input space is divided into nine different regions. In each region, an explicit expression of $F(x_1,x_2)$ can be derived. The resulting expressions are shown in Table 3.

Table 3. The explicit expression of $F(x_1,x_2)$ derived for the nine different regions that divide up the whole input space, i.e. the $x_1(n)$–$x_2(n)$ plane, shown in Fig. 2

Region no.	$F(x_1,x_2)$
1	Expression (16)
2	$k_1[(1 - k_2)x_2(n) + (1 + k_2)L]/2L$
3	k_1
4	$k_1[(1 - k_3)x_1(n) + (1 + k_3)L]/2L$
5	k_1k_3
6	$k_1[(k_3 - k_4)x_2(n) + (k_3 + k_4)L]/2L$
7	k_1k_4
8	$k_1[(k_2 - k_4)x_1(n) + (k_2 + k_4)L]/2L$
9	k_1k_2

fuzzy control systems operate in this region most of the time as $(x_1, x_2) = (0, 0)$ is the system equilibrium point. The value of L affects the overall control performance. Too small a value of L will make Region 1 too small, which may force the system to stay outside Region 1 too often during transition. This could adversely affect the control performance. Too large a value of L will do the opposite. An appropriate value of L should be selected.

Some important characteristics of $F(x_1, x_2)$ for Region 1 were observed:

(1) The characteristics of $F(x_1, x_2)$ were parameterized by k_j. When symmetrical fuzzy control rules were used (symmetry in terms of input state, that is, when $R_1 = R_4$ and $R_2 = R_3$, or equivalently, when $k_4 = k_1 = 1$ and $k_2 = k_3$), the $x_1(n)$ and $x_2(n)$ terms of $F(x_1, x_2)$ would not exist and only the $x_1(n)x_2(n)$ and L^2 terms remained. When asymmetrical control rules were employed (i.e. $k_4 \neq 1$ or $k_2 \neq k_3$), either the $x_1(n)$ term or $x_2(n)$ term remained and $F(x_1, x_2)$ was asymmetric.

(2) At the equilibrium point $(0, 0)$, $F(0, 0) = k_1(1 + k_2 + k_3 + k_4)/4$. The value of $F(x_1, x_2)$ at the four corners of Region 1 was: $F(L, L) = k_1$, $F(-L, L) = k_1k_3$, $F(-L, -L) = k_1k_4$ and $F(L, -L) = k_1k_2$.

(3) Once the value of L was determined, there were three unknown parameters left in (16), namely k_2, k_3 and k_4. There were four different terms: L^2, $x_1(n)$, $x_2(n)$ and $x_1(n)x_2(n)$. The coefficients of these terms were determined by k_2, k_3 and k_4. Since the number of equations was more than the number of unknown parameters, k_2, k_3 and k_4 might not exist to produce some given coefficients. On the other hand, given the values of k_2, k_3 and k_4, the coefficients were always computable.

(4) Whether the gain variation was sensible in the context of control depended on the values of k_j. The values of k_j had to be carefully selected.

(5) $F(x_1, x_2)$ was a continuous function for any values of $x_1(n)$ and $x_2(n)$. The continuity always held on the boundaries of any adjacent regions.

We now present the characteristics of $F(x_1, x_2)$ when k_j are of some specific values. The aims were (1) to use these values as examples to reveal useful characteristics of $F(x_1, x_2)$ in the context of controller gain variation; (2) to demonstrate that the gain variation empowered the nonlinear PI controllers to outperform their linear counterparts; and (3) to illustrate how to design $F(x_1, x_2)$ by properly selecting k_j values to achieve desired gain variation characteristics.

3.3.2. Characteristic of the variable gains of the nonlinear PI controllers when $k_1 = 1$, $k_2 = k_3 = 0$ and $k_4 = 1$. When $k_1 = 1$, $k_2 = k_3 = 0$ and $k_4 = 1$, the four simplified linear TS rules in Table 1 became the four symmetrical linear rules in Table 4. We chose these particular k_j values because they resulted in some of the simplest useful control rules. The meaning of these four rules could be linguistically interpreted as follows. When the process output is below the setpoint but the process output is increasing to approach the setpoint (R_2) or when the process output is above the setpoint but the process output is decreasing to reach the setpoint (R_3), the fuzzy controller output should be unchanged (i.e. $\Delta u(n) = 0$). That is, the consequent in R_2 and R_3 should be zero. When the process output is below the setpoint and is still decreasing, R_1 should try to generate a positive $v_1(n)$ to increase the controller output for driving the process output to the setpoint. Conversely, when the process output is above the setpoint and is still increasing, R_4 should attempt to produce a negative $v_4(n)$ to reduce the controller output, pulling the process output back to the setpoint.

Table 4. The resulting four simplified linear TS fuzzy control rules when $k_1 = k_4 = 1$ and $k_2 = k_3 = 0$ in Table 1

Rule no.	IF $x_1(n)$ is	AND $x_2(n)$ is	THEN
R_1	Positive	Positive	$v_1(n) = a_1x_1(n) + a_2x_2(n)$
R_2	Positive	Negative	$v_2(n) = 0$
R_3	Negative	Positive	$v_3(n) = 0$
R_4	Negative	Negative	$v_4(n) = a_1x_1(n) + a_2x_2(n)$

Substituting $k_1 = k_4 = 1$ and $k_2 = k_3 = 0$ in Table 3, we obtained $F(x_1, x_2)$ for the nine regions, shown in Fig. 2, as listed in Table 5. We also plot the corresponding $F(x_1, x_2)$ in Fig. 3 for visualization. Without loss of generality, we assume $L = 1$ and plot $F(x_1, x_2)$ with respect to $x_1(n)$ and $x_2(n)$ whose ranges are $[-2L, 2L]$. Assuming $L = 1$ is not restrictive as one can always use scalars to scale the input variables to fit any given intervals and the scaled input variables can then be treated as $x_1(n)$ and $x_2(n)$. As pointed out above, Region 1 (that is, $x_1(n)$ and $x_2(n)$ are in the square area of $[-L, L] \times [-L, L]$) is of most importance. In the rest of this subsection, we will confine our analysis of $F(x_1, x_2)$ to this region.

$F(x_1, x_2)$ is a symmetric function with respect to the lines $x_1(n) = x_2(n)$ and $x_1(n) = -x_2(n)$. This means that the proportional gain and integral gain vary symmetrically in terms of these two lines, resulting in symmetrical PI control for symmetric input states. As an illustration, if $\Delta u(n)$ is W_1 and W_2 at (X_1, X_2) and $(-X_1, X_2)$ respectively, then $\Delta u(n)$ is W_1 and W_2 respectively at $(-X_1, -X_2)$ and $(X_1, -X_2)$. The continuity of $F(x_1, x_2)$ can also be visualized conveniently in Fig. 3.

At $(0, 0)$, $F(x_1, x_2) = 0.5$. $F(x_1, x_2)$ reached its maximum, 1, at (L, L) and $(-L, -L)$ and achieved its minimum, 0, at $(L, -L)$ and $(-L, L)$. We defined the proportional gain and integral gain in (15) at

Table 5. The resulting explicit expression of $F(x_1, x_2)$ when $k_1 = k_4 = 1$ and $k_2 = k_3 = 0$ in Table 3

Region no.	$F(x_1, x_2)$
1	$(x_1(n)x_2(n) + L^2)/2L^2$
2	$(x_2(n) + L)/2L$
3	1
4	$(x_1(n) + L)/2L$
5	0
6	$(-x_2(n) + L)/2L$
7	1
8	$(-x_1(n) + L)/2L$
9	0

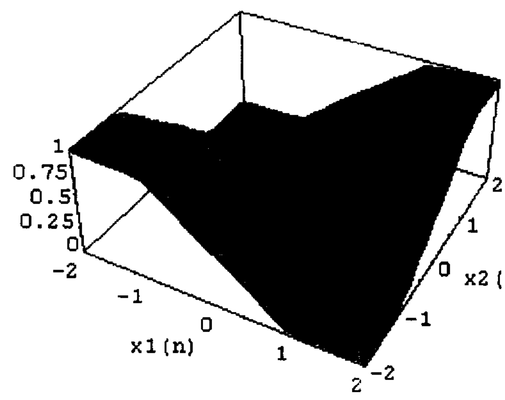

Fig. 3. A three-dimensional plot of $F(x_1, x_2)$ whose expressions corresponding to the nine different regions in the $x_1(n)$–$x_2(n)$ plane are given in Table 5. Without loss of generality, L is assumed to be 1, and $x_1(n)$ and $x_2(n) \in [-2L, 2L]$. $F(x_1, x_2)$ is most nonlinear and of most interest and importance when $x_1(n)$ and $x_2(n)$ are in the square area of $[-L, L] \times [-L, L]$.

(0, 0) as steady-state proportional gain (denoted as $K_p(0,0)$) and integral-gain (denoted as $K_i(0,0)$), respectively. The gains at any other states were named dynamic proportional gain (denoted as $K_p(x_1, x_2)$) and integral gain (denoted as $K_i(x_1, x_2)$). Based on the maximum and minimum of $F(x_1, x_2)$, one sees that the dynamic gains could be enlarged to as much as two times of the steady-state gains or could be reduced to as little as zero.

We now study the gain variation in the context of control and in comparison with the gains of the corresponding linear PI controller. Since the gains of the linear PI controller do not vary, to make a fair comparison, we made the steady-state gains of the nonlinear PI controllers equal to those of the linear PI controller. Specifically, we let $a_2 = 2\bar{K}_p$ and $a_1 = 2\bar{K}_i$ so that $K_p(0,0) = \bar{K}_p$ and $\bar{K}_i(0,0) = \bar{K}_i$ (K_p and $\bar{K}_i$ are defined in (8)).

When both $x_1(n)$ and $x_2(n)$ are in the first or third quadrant of the $x_1(n)$–$x_2(n)$ plane (excluding the origin) where the product $x_1(n)x_2(n)$ is positive, according to $F(x_1, x_2)$ for Region 1 in Table 5,

$$K_p(x_1, x_2) > K_p(0,0) = \bar{K}_p$$

and

$$K_i(x_1, x_2) > K_i(0,0) = \bar{K}_i.$$

Roughly speaking, the farther the current input state $(x_1(n), x_2(n))$ is away from (0, 0), the larger $K_p(x_1, x_2)$ and $K_i(x_1, x_2)$ are as compared to $K_p(0,0)$ and $K_i(0,0)$, respectively. This statement becomes strictly true when $(x_1(n), x_2(n))$ is on the line $x_1(n) = x_2(n)$. The maximal gains are reached at (L, L) or $(-L, -L)$ and are kept as such for either $x_1(n) \geq L$ and $x_2(n) \geq L$ or $x_1(n) \leq -L$ and $x_2(n) \leq -L$. In these two quadrants, the process output is in one of the following two situations: (1) the process output is already above the setpoint and is still increasing; or (2) the process output is below the setpoint and is still decreasing. In either case, the process output is running away from the setpoint. From control point of view, a larger decrement of the controller output (or increment for the second situation) is beneficial. The farther the process output is away from the setpoint and/or the faster the process output is running away from the setpoint, the greater the decrement (or increment) should be so that the process output is driven to the setpoint more quickly. On the other hand, when $(x_1(n), x_2(n))$ is close to (0, 0), a smaller decrement or increment of the controller output is advantageous in order to avoid possible system instability due to excessive change of the controller output. The gain variation of the fuzzy controllers clearly implements these control strategies naturally and inherently. In these two quadrants, the signs of $x_1(n)$ and $x_2(n)$ are the same. The proportional control and integral control are additive in absolute value term. The variable gains provide variable amplification of the addition outcome.

When both $x_1(n)$ and $x_2(n)$ are in the second or fourth quadrant of the $x_1(n)$–$x_2(n)$ plane (excluding the origin) where the product $x_1(n)x_2(n)$ is negative,

$$K_p(x_1, x_2) < K_p(0,0) = \bar{K}_p$$

and

$$\bar{K}_i(x_1, x_2) < K_i(0,0) = \bar{K}_i.$$

Approximately speaking, the farther $(x_1(n), x_2(n))$ is away from (0, 0), the smaller $K_p(x_1, x_2)$ and $K_i(x_1, x_2)$ are compared to $K_p(0,0)$ and $K_i(0,0)$, respectively. The statement becomes precise if $(x_1(n), x_2(n))$ is on the line $x_1(n) = -x_2(n)$. The minimal gains, 0, are achieved at $(-L, L)$ or $(L, -L)$ and remain as such for either $x_1(n) \geq L$ and $x_2(n) \leq L$ or $x_1(n) \leq -L$ and $x_2(n) \geq L$. In these two quadrants, the process output is in one of the two situations: (1) the process output is above the setpoint but is decreasing; or (2) the process output is below the setpoint but is increasing. In either situation, the process output is approaching the setpoint. Because of this, the gains are reduced to avoid excessive change of the controller output that could result in unwanted oscillation of the process output around the setpoint. The faster the process output is approaching the setpoint, the more the gain reduction. The gains become zero if the approaching rate is too big (i.e. when $x_2(n) \le -L$ or $x_2(n) \ge L$), regardless of the value of $x_1(n)$. Zero gain makes the controller output unchanged (that is, $\Delta u(n) = 0$). The gain reduction is insignificant if the approaching rate of the process output is small and the process output is close to the equilibrium point. Since the signs of $x_1(n)$ and $x_2(n)$ are opposite, the proportional control and integral control are subtractive in absolute value term. The gain variation makes the subtraction result smaller.

The above analysis of the variable gains characteristics reveals that the gain variation is desirable because it enables the fuzzy controllers to perform better than the linear PI controller. This assessment could further be substantiated by computer simulation, just as we did in our previous paper (Ying *et al.*, 1990) where a similar but different nonlinear variable gain PI controller was shown to outperform its corresponding linear PI controller when controlling a nonlinear process as well as a linear first-order process with time delay. The gain variation had little advantage when controlling the linear processes. As our analysis had provided the insightful reasons for the assessment, for brevity, we did not include computer simulation in this paper.

3.3.3. Stability of the nonlinear variable gain PI control systems. As with any type of control system, system stability involving the nonlinear variable gain PI controllers is an important issue. In our previous paper, in order to show that some Mamdani fuzzy controllers were nonlinear variable gain PI controllers (Ying *et al.*, 1990; Ying, 1993a) (they are different from those in this paper, though), we studied local stability of those fuzzy control systems around the system equilibrium point (0, 0). The idea was that the nonlinear PI controllers could be linearized around the equilibrium point and the linearized controllers became the linear PI controller. Thus, if the corresponding linear PI control system was stable (or unstable) at (0, 0), the nonlinear PI control system was also stable (or unstable) at (0, 0) (see Slotine and Li (1991) for the basis). We also studied the global stability of some of those nonlinear variable gain PI control systems using classical stability analysis tools such as the Small Gain Theorem (Chen and Ying, 1998). The stability analysis presented in our above-mentioned papers are fully applicable to the local and global stability analysis of the fuzzy control systems involving the nonlinear variable gain PI controllers presented in this paper.

3.3.4. Obtaining desired gain variation characteristics and avoiding insensible ones through appropriate selection of k_j values—two examples. Based on the general and specific characteristics of the variable gains in relation to k_j values presented in Sections 3.3.1 and 3.3.2, it is possible for one to obtain the desired gain variation characteristics by choosing proper values of k_j. Recall that $F(0,0) = k_1(1 + k_2 + k_3 + k_4)/4$, $F(L,L) = k_1$, $F(-L,L) = k_1k_3$, $F(-L,-L) = k_1k_4$ and $F(L,-L) = k_1k_2$. The key is to select such k_j values that $F(0,0)$, $F(L,L)$, $F(-L,L)$, $F(-L,-L)$ and $F(L,-L)$ meet pre-specified conditions, as these five values define some major characteristics of $F(x_1, x_2)$.

An example to prove this point is the feedback control of mean arterial pressure in postsurgical patients at Cardiac Surgical Intensive Care Unit where some patients exhibit hypertension due to heart surgeries (Sheppard, 1980; Ying *et al.*, 1992). The blood pressure is lowered and maintained at a normal level (usually 80 mmHg) through regulating infusion rate of a vasodilating drug, called sodium nitropresside, that is delivered to the patient intravenously. Increasing the infusion rate lowers the blood pressure further while decreasing the rate creates an opposite effect on the blood pressure. Because a low blood pressure, say 50 mmHg, is far more life-threatening than a high blood pressure, say 110 mmHg, biased change of the infusion rate must be implemented. The infusion rate must be reduced much more quickly when the blood pressure is quite below the normal level. The farther the blood pressure is below the normal level, the faster the rate reduction must be. The first automatic blood pressure controller implemented clinically in 1970's (Sheppard, 1980) indeed developed this kind of biased control strategies. Specifically, the controller was a PI controller with a decision table that had seven rules for changing the proportional gain and integral gain according to the current state of the blood pressure. One of the rules doubled the controller gains whenever the blood pressure became 5 mmHg below the normal level, trying to speed up the reduction of the infusion rate.

With our nonlinear variable gain PI controllers, it is possible to realize these biased control strategies in a more smooth and natural fashion. We first let $k_2 = k_3 = 0$ and the meaning of such selection in the context of control is given in Section 3.3.2. We then let k_4 be less than k_1. Because $k_1 = 1$, $k_4 < 1$, making $F(L,L)$ greater than $F(-L,-L)$. As a consequence, $F(x_1, x_2)$ in the first quadrant of the $x_1(n)$–$x_2(n)$ plane is, roughly and globally speaking, bigger than that in the third quadrant, making the rate reduction faster than rate increase. Numerically, let $k_4 = 1/\lambda$ where λ is a constant and $\lambda > 1$. Hence, $F(0,0) = (1 + \lambda)/4\lambda$. Different value of λ can create different degree of biased control. If $\lambda = 3$ (other reasonable λ values such as 2 or 4 are also fine), $k_4 = 1/3$, $F(-L, -L) = k_4 = 1/3$ and $F(0,0) = 1/3$. This means that (1) the controller gains at (L, L) are three times as large as those at $(0, 0)$ or $(-L, -L)$; and (2) $F(x_1, x_2)$ in the third quadrant is quite "flat" (because $F(0,0) = F(-L, -L)$), meaning the gain variation in that quadrant is small. This $F(x_1, x_2)$ is plotted in Fig. 4. One sees that the biased control strategies are indeed realized as the gain variation in the first quadrant is much bigger and steeper than those in the third quadrant, as desired.

Inappropriate values of k_j's can produce irrational $F(x_1, x_2)$ and hence illogical and unreasonable gain variation from standpoint of control. For instance, if the values of k_j's are chosen such that

$$1 + k_2 + k_3 + k_4 = 0,$$

$F(x_1, x_2)$ for Region 1 becomes

$$\begin{aligned} F(x_1, x_2) = \frac{k_1}{4L^2}[&(1 + k_2 - k_3 - k_4)Lx_1(n) \\ &+ (1 - k_2 + k_3 - k_4)Lx_2(n) \\ &+ (1 - k_2 - k_3 + k_4)x_1(n)x_2(n)]. \end{aligned}$$

Obviously, $F(0,0) = 0$, meaning the proportional gain and integral gain at the equilibrium point are zero. This class of $F(x_1, x_2)$ is not usable as far as control is concerned. We plot one of such $F(x_1, x_2)$ in Fig. 5 for graphical presentation where we let $k_1 = 1$, $k_2 = k_3 = 0$ and $k_4 = -1$. This example highlights the importance of the selection of proper k_j's values.

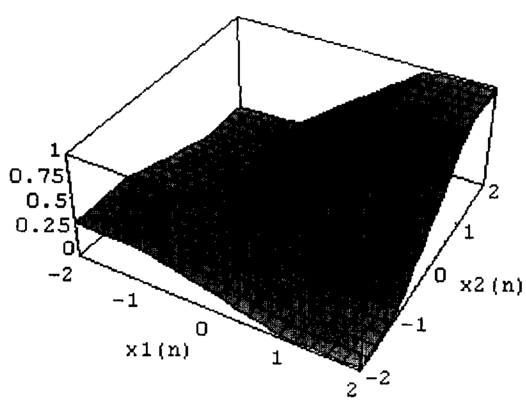

Fig. 4. A three-dimensional plot of $F(x_1, x_2)$ when $k_1 = 1$, $k_2 = k_3 = 0$ and $k_4 = 1/3$. Without loss of generality, L is assumed to be 1, and $x_1(n)$ and $x_2(n) \in [-2L, 2L]$. $F(x_1, x_2)$ is most nonlinear and of most importance when $x_1(n)$ and $x_2(n)$ are in the square area of $[-L, L] \times [-L, L]$. $F(x_1, x_2)$ at (L, L) are three times as large as those at $(0, 0)$ or $(-L, -L)$, meaning the PI controller gains can be up to three times larger in the first quadrant of the $x_1(n)$–$x_2(n)$ plane than at $(0, 0)$ or $(-L, -L)$. $F(x_1, x_2)$ in the third quadrant is quite "flat" (note $F(0,0) = F(-L, -L)$), meaning the gain variation is insignificant).

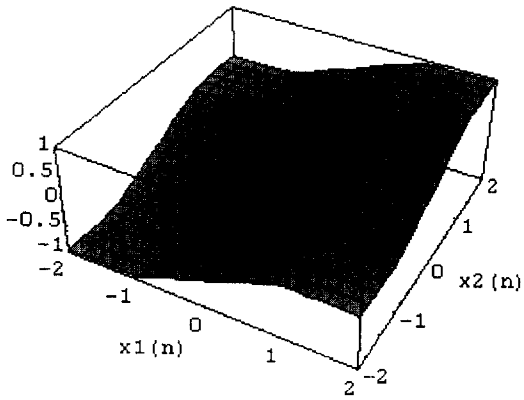

Fig. 5. A three-dimensional plot of $F(x_1, x_2)$ when $k_1 = 1$, $k_2 = k_3 = 0$ and $k_4 = -1$. Without loss of generality, L is assumed to be 1, and $x_1(n)$ and $x_2(n) \in [-2L, 2L]$. The plot shows that inappropriate values of k_j can lead to unreasonable and illogical gain variation and hence an unusable controller. Specifically in this example, $F(0,0) = 0$, resulting in zero gains of the nonlinear PI controllers at the equilibrium point.

4. CONCLUSIONS

In this paper, we have introduced a simplified TS fuzzy rule scheme that greatly reduces the number of unknown adjustable parameters required by the original TS fuzzy rule scheme. This advantage is important as it makes the original TS rule scheme more efficient. We have theoretically proved that the TS fuzzy controllers using the simplified linear TS rules are nonlinear variable gain controllers. The characteristics of the gain variation are parameterized by the proportional factors, namely k_j, in the simplified rules. As special cases, the fuzzy controllers with three or two input variables become nonlinear variable gain PID, PI or PD controllers. We have also provided an in-depth analysis of the nonlinear variable gain PI controllers, and have shown how to achieve the desired gain variation characteristics and how to identify unreasonable ones. The gain variation empowers the nonlinear variable gain PI controllers to outperform their linear counterpart when controlling nonlinear processes. The methods and principles developed are applicable to the other TS fuzzy controllers that use the simplified TS rules.

These insightful analysis and results, being the first ones available, bridge the wide knowledge gap existing in the current literature regarding the analytical structure of the TS fuzzy controllers as well as their possible connection with conventional controllers, including the PID controller. With these results, one can proceed to investigate many important issues such as fuzzy control system stability and robustness.

Finally, compared to the nonlinear variable gain PI controllers derived from some Mamdani fuzzy controllers in our earlier study (Ying *et al.*, 1990; Ying, 1993a), the nonlinear PI controllers derived in this paper are vastly more diverse in gain variation characteristics. This is directly owing to the use of our simplified TS rule scheme that parameterizes the characteristics of the gain variation. As a result, there are an infinitely large number of different gain variation characteristics. In sharp contrast, each of the nonlinear PI controllers in (Ying *et al.*, 1990; Ying, 1993a) can generate only one. We conclude that the nonlinear PI controllers studied in this paper are potentially capable of offering more and better solutions to a wider variety of nonlinear control problems. All these nonlinear variable gain PI, PD and PID controllers can offer superior nonlinear control to the popular linear PI, PD and PID controllers.

Acknowledgements—This work was supported in part by grant No. 004952-054 from Texas Higher Education Coordinating Board.

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