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Brief Paper

A Nonlinear Fuzzy Controller with Linear Control Rules is the Sum of a Global Two-dimensional Multilevel Relay and a Local Nonlinear Proportional-integral Controller*

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Key Words—Control system analysis; fuzzy control; nonlinear control systems; PID control; relay control.

Abstract—The author analytically proves that a nonlinear fuzzy controller with linear control rules and N members for input fuzzy sets is the sum of a global two-dimensional multilevel relay and a local nonlinear proportional-integral (PI) controller which adjusts the control action generated by the global multilevel relay. As N increases, the resolution of the global multilevel relay is enhanced but the role of the local nonlinear PI controller in total control action is decreased. As N approaches ∞, the global multilevel relay approaches a regular linear PI controller while the control action from the local nonlinear PI controller approaches zero. The role of the global multilevel relay and the local nonlinear PI controller in total control action is quantitatively described, as is the degree of nonlinearity of the fuzzy controllers with different N.

1. *Introduction*

TO ADVANCE FUZZY control technique, sound theory needs to be developed. The author believes that one way to develop such theory is to analytically investigate structures of fuzzy controllers and relate the structures to nonfuzzy control theory. Such relations will provide solid frameworks for analytically solving many important but previously difficult problems in fuzzy control technique, such as stability, by utilizing abundant well-developed and powerful nonfuzzy control techniques.

To reveal structures of fuzzy controllers and link the structures with nonfuzzy control theory, a novel method was initially developed (Ying, 1987), presented (Ying *et al.*, 1988) and published (Ying *et al.*, 1990). The work showed that the simplest possible nonlinear fuzzy controller with two members for the input fuzzy sets, "error" and "rate change of error" ("rate" for short) was equivalent to a regular linear PI controller when a linear defuzzification algorithm was used or to a nonlinear PI controller when a nonlinear defuzzification algorithm was used. Using this method, the results on the linear properties of the fuzzy controller were generalized to fuzzy controllers with more members for the input fuzzy sets and different fuzzy logic, first by Siler and Ying (1989) and then by Buckley and Ying (1990) and Buckley (1989a). Moreover, the Limit Theorems for linear fuzzy control rules were developed (Buckley and Ying, 1989) and extended to multiple-input–multiple-output fuzzy controllers (Buckley, 1990). Following the generalization of the results on the linear properties, the results on the nonlinear aspects of the fuzzy controller were also mathematically generalized to the fuzzy controller with more members for the input fuzzy sets, first by Buckley (1989b) and then by Wang *et al.* (1990).

In this paper, a nonlinear fuzzy controller with linear control rules is first defined. The author then analytically derives the explicit structure of the fuzzy controller and relates the resultant structure to the multilevel relay and PI controller of nonfuzzy control theory.

2. *Theoretical analysis of structure of the nonlinear fuzzy controller*

2.1. *Components of the nonlinear fuzzy controller.* If T denotes sampling period and nT (n is a positive integer) denotes sampling time, then the scaled inputs at sampling time nT are

$$e^* = GE \cdot e(nT) = GE[y(nT) - \text{setpoint}], \qquad (2.1)$$

$$r^* = GR \cdot r(nT) = GR[e(nT) - e(nT - T)], \qquad (2.2)$$

where $e(nT)$, $r(nT)$ and $y(nT)$ designate crisp unscaled error, rate, and process output at sampling time nT, respectively, and $e(nT - T)$ specifies crisp unscaled error at sampling time $(n-1)T$. GE and GR are the scalars for the crisp error and rate.

Let the number of members of the fuzzy sets "error" and "rate" be the same and the membership functions be identical. This condition can easily be met, since if the number of members differs, some members can be added to the smaller set to attain equality. Assume there are $J (J \geq 1)$ members for positive "error" ("rate"), J members for negative "error" ("rate") and one member for zero "error" ("rate"). Therefore, there are a total of

$$N = 2J + 1 \geq 3, \qquad (2.3)$$

members for the fuzzy set "error" ("rate"). Index systems

$$\{E_{-J}, E_{-J+1}, \ldots, E_{-1}, E_0, E_1, \ldots, E_{J-1}, E_J\},$$

and

$$\{R_{-J}, R_{-J+1}, \ldots, R_{-1}, R_0, R_1, \ldots, R_{J-1}, R_J\}, \qquad (2.4)$$

are adopted to establish relationships between the indexes and the names of the members of the fuzzy sets "error" and "rate". E_i represents a member of the fuzzy set "error" and R_i represents a member of the fuzzy set "rate". The positive indexes specify the members for positive error (rate), the negative indexes denote the members for negative error (rate) and the index 0 corresponds to the zero error (rate) of the fuzzy sets. The membership functions corresponding to

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the members in (2.4) are expressed as:

$$\{\mu_{-J}(x),\ \mu_{-J+1}(x),\ \ldots,\mu_{-1}(x),\ \mu_0(x),\ \mu_1(x),\ \ldots,\ \mu_{J-1}(x),\ \mu_J(x)\}. \quad (2.5)$$

Denote the central value of the membership function $\mu_i(x)$ as λ_i and define $\lambda_{-J}=-L$, $\lambda_0=0$, and $\lambda_J=L$. Also, let the space between the central values of two adjacent members be equal. Then the space S is:

$$S=\frac{L}{J}, \quad (2.6)$$

and consequently the central value of $\mu_i(x)$ is $\lambda_i=i\cdot S$. It is obvious that the base of each member is $2S$. It should be noted that the equality of the bases does not imply loss of generality because the bases of the members of "error" and "rate" are different with respect to the actual unscaled inputs, $e(nT)$ and $r(nT)$.

The membership function $\mu_i(x)$ in this study is the commonly-used triangular-shaped membership function satisfying the following conditions:

(1) For $i=-J+1,\ -J+2,\ \ldots,\ J-2,\ J-1$,

$$\mu_i(x)=\frac{1}{S}[x-(i-1)S], \quad \text{if } x\in[(i-1)S,\ iS],$$

$$\mu_i(x)=-\frac{1}{S}[x-(i+1)S], \quad \text{if } x\in[iS,\ (i+1)S],$$

$$\mu_i(x)=0, \quad \text{if } x\notin[(i-1)S,\ (i+1)S]$$

(2) For $i=J$ or $i=-J$,

$$\mu_J(x)=\frac{1}{S}[x-(J-1)S], \quad \text{if } x\in[(J-1)S,\ JS],$$

$$\mu_J(x)=1, \quad \text{if } x\in[JS,\ +\infty)$$
$$\text{and } \mu_J(x)=0, \quad \text{if } x\notin[(J-1)S,\ +\infty),$$

$$\mu_{-J}(x)=-\frac{1}{S}[x-(-J+1)S], \quad \text{if } x\in[-JS,\ (-J+1)S],$$

$$\mu_{-J}(x)=1, \quad \text{if } x\in(-\infty,\ JS]$$
$$\text{and } \mu_{-J}(x)=0, \quad \text{if } x\notin(-\infty,\ (-J+1)S].$$

It is obvious that

$$\mu_i(x)+\mu_{i+1}(x)=1, \quad x\in(-\infty,\ +\infty). \quad (2.7)$$

Figure 1 shows an example of such a membership function with $N=7$ ($J=3$) and $S=5$. In this paper, $\mu_i(e^*)$ is denoted as the membership for E_i and $\mu_i(r^*)$ as the membership for R_i.

Assume there are $2N-1$ (or $4J+1$) members in the fuzzy set "output". Among these, $2J$ members are for positive "output," $2J$ members are for negative "output" and one member is for zero "output". Using the index system (2.5), the members of the fuzzy set "output" can be described by

$$\{U_{-2J},\ U_{-2J+1},\ \ldots,\ U_{-1},\ U_0,\ U_1,\ \ldots,\ U_{2J-1},\ U_{2J}\}. \quad (2.8)$$

The central values of the members of the fuzzy set "output" are designated as γ_i. Let $\gamma_{-2J}=-H$, $\gamma_0=0$ and $\gamma_{2J}=H$. Further, let the space V between the central values of two adjacent members be equal. Therefore the space is

$$V=\frac{H}{2J}=\frac{H}{N-1}, \quad (2.9)$$

and the ith central value can be written as

$$\gamma_i=i\cdot V=\frac{i\cdot H}{N-1}. \quad (2.10)$$

For the fuzzy set "output," the author requires: (1) the membership function be symmetrical about its central value; and (2) the shape of the membership functions of all the members be the same.

It is necessary to use N^2 fuzzy control rules to cover all the possible combinations of N members of the fuzzy set "error" and N members of the fuzzy set "rate". In the study, the fuzzy control rules must comply with the following rule:

IF "error" is E_i and "rate" is R_j THEN "output" is $U_{-(i+j)}$. (2.11)

In other words, the index of the member of "output" is always equal to the negative sum of the indexes of the members of "error" and the members of "rate". Such a control rule is called here a linear control rule. To illustrate this rule, take $N=5$ as an example. If the members of the input fuzzy sets are {negative medium (NM), negative small (NS), zero (ZO), positive small (PS), positive medium (PM)} and are represented as $\{E_{-2},\ E_{-1},\ E_0,\ E_1,\ E_2\}$ and $\{R_{-2},\ R_{-1},\ R_0,\ R_1,\ R_2\}$, then the corresponding nine members of the fuzzy set "output" can be indexed as $\{U_{-4},\ U_{-3},\ U_{-2},\ U_{-1},\ U_0,\ U_1,\ U_2,\ U_3,\ U_4\}$ which may be interpreted as {negative very large (NVL), negative large

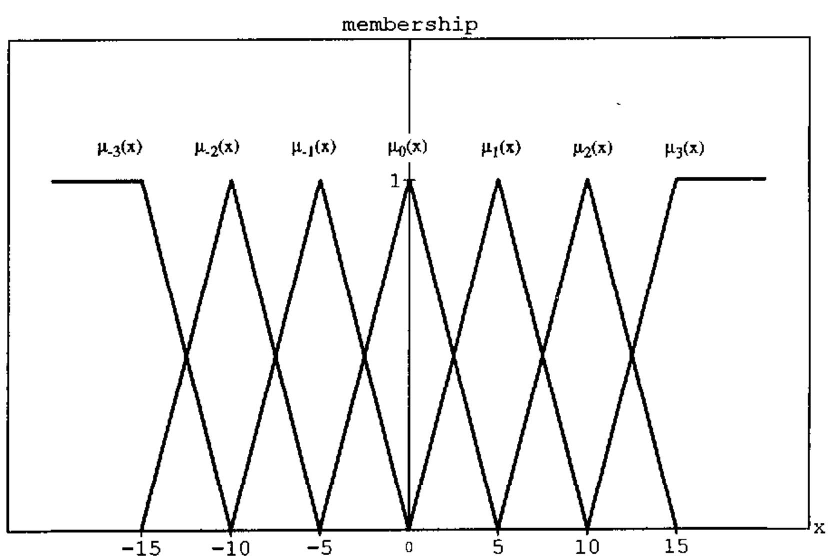

FIG. 1. An example of a triangular-shaped membership function with seven members ($N=7$). The space between the central value of two adjacent members is 5 ($S=5$).

TABLE 1. AN EXAMPLE TO SHOW HOW TO CONSTRUCT 25 FUZZY CONTROL RULES ACCORDING TO THE RULE (2.11) WHEN $N = 5$

	E_{-2}(NM)	E_{-1}(NS)	E_0(ZO)	E_1(PS)	E_2(PM)
R_{-2}(NM)	U_4(PVL)	U_3(PL)	U_2(PM)	U_1(PS)	U_0(ZO)
R_{-1}(NS)	U_3(PL)	U_2(PM)	U_1(PS)	U_0(ZO)	U_{-1}(NS)
R_0(ZO)	U_2(PM)	U_1(PS)	U_0(ZO)	U_{-1}(NS)	U_{-2}(NM)
R_1(PS)	U_1(PS)	U_0(ZO)	U_{-1}(NS)	U_{-2}(NM)	U_{-3}(NL)
R_2(PM)	U_0(ZO)	U_{-1}(NS)	U_{-2}(NM)	U_{-3}(NL)	U_{-4}(NVL)

(NL), NM, NS, ZO, PS, PM, positive large (PL), positive very large (PVL)}. The corresponding 25 fuzzy control rules satisfying the rule (2.11) are shown in Table 1.

Zadeh fuzzy logic AND is used to execute the IF side of the fuzzy control rule. That is

$$\mu(i, j) = \text{Min}\,(\mu_i(e^*), \mu_j(r^*)), \tag{2.12}$$

where $\mu(i, j)$ is the membership of the member of the fuzzy set "output" obtained when E_i and R_j are used in the IF side. Because the membership function of "output" is symmetrical about its central value, the central value of the member $U_{-(i+j)}$, $\gamma_{-(i+j)}$, and the resultant membership from the IF side, namely $\mu(i, j)$, are used to calculate the THEN side of the fuzzy control rule, i.e.

$$v(i, j) = \mu(i, j) \cdot \gamma_{-(i+j)} = -\text{Min}\,(\mu_i(e^*), \mu_j(r^*)) \cdot (i + j)V, \tag{2.13}$$

where $v(i, j)$ is the incremental control output contributed by the fuzzy control rule (2.11). If more than one membership results, say μ_1 and μ_2, from executing two different fuzzy control rules, Lukasiewicz fuzzy logic OR is used to get combined membership, μ, because the conditions being ORed are maximally negatively correlated. That is

$$\mu = \text{Min}\,(\mu_1 + \mu_2, 1). \tag{2.14}$$

Recall that the shapes of the membership functions of "output" were required to be the same. In the defuzzification process, therefore, the contribution from the members of "output" in the THEN side of the fuzzy control rules should be weighted by their memberships calculated from the IF side. Consequently, the scaled crisp incremental output, $GU \cdot \Delta u(nT)$, can be calculated by the following defuzzification algorithm

$$GU \cdot \Delta u(nT) = GU \frac{\sum_{v(i,j)\neq 0} v(i, j)}{\sum_{\mu(i,j)\neq 0} \mu(i, j)} = GU \frac{\sum_{\mu(i,j)\neq 0} \mu(i, j)\gamma_{-(i+j)}}{\sum_{\mu(i,j)\neq 0} \mu(i, j)}. \tag{2.15}$$

Finally, a new crisp output of the fuzzy controller at sampling time nT is calculated as

$$u(nT) = u(nT - T) + GU \cdot \Delta u(nT), \tag{2.16}$$

where GU is the scalar for incremental output and $u(nT - T)$ is the output of the fuzzy controller at sampling time $(n - 1)T$.

2.2. *Analytical analysis of structure of the nonlinear fuzzy controller with linear control rules.*

Theorem 1. The structure of the nonlinear fuzzy controller with linear control rules is the sum of a global two-dimensional multilevel relay and a local nonlinear PI controller.

Proof. The author first proves the theorem in the situations in which both e^* and r^* are within the interval $[-L, L]$. Others situations will be dealt with later.

(A) Both e^* and r^* are within the interval $[-L, L]$. With losing generality, assume that

$$iS \le e^* \le (i + 1)S, \quad jS \le r^* \le (j + 1)S. \tag{2.17}$$

Being fuzzified, the memberships of e^* and r^* are obtained as

$$\mu_i(e^*) = -\frac{1}{S}[e^* - (i + 1)S], \quad \mu_{i+1}(e^*) = \frac{1}{S}[e^* - iS], \tag{2.18}$$

$$\mu_j(r^*) = -\frac{1}{S}[r^* - (j + 1)S], \quad \mu_{j+1}(r^*) = \frac{1}{S}[r^* - jS], \tag{2.19}$$

which are the memberships for the members E_i, E_{i+1}, R_j and R_{j+1}, respectively. Membership for all other members of "error" and "rate" is zero. Therefore, only the following four fuzzy control rules are executed:

(r1) If "error" is E_{i+1} and "rate" is R_{j+1} then "output" is $U_{-(i+j+2)}$

(r2) If "error" is E_{i+1} and "rate" is R_j then "output" is $U_{-(i+j+1)}$

(r3) If "error" is E_i and "rate" is R_{j+1} then "output" is $U_{-(i+j+1)}$.

(r4) If "error" is E_i and "rate" is R_j then "output" is $U_{-(i+j)}$.

Applying equation (2.13) to each of the fuzzy control rules results in the following:

(r1*) $v(i + 1, j + 1) = -\text{Min}\,(\mu_{i+1}(e^*), \mu_{j+1}(r^*)) \cdot (i + j + 2)V$

(r2*) $v(i + 1, j) = -\text{Min}\,(\mu_{i+1}(e^*), \mu_j(r^*)) \cdot (i + j + 1)V$

(r3*) $v(i, j + 1) = -\text{Min}\,(\mu_i(e^*), \mu_{j+1}(r^*)) \cdot (i + j + 1)V$

(r4*) $v(i, j) = -\text{Min}\,(\mu_i(e^*), \mu_j(r^*)) \cdot (i + j)V$.

To determine the results of the Min operations in (r1*) to (r4*), the author configures a square by the intervals $[iS, (i + 1)S]$ and $[jS, (j + 1)S]$ and divides the square into eight regions as shown in Fig. 2. In different regions, $\mu_i(e^*)$, $\mu_{i+1}(e^*)$, $\mu_j(r^*)$ and $\mu_{j+1}(r^*)$ have different relationships in terms of the magnitudes of the memberships. The outcomes of evaluating the Min operations are illustrated in Table 2. Since the fuzzy control rules r2 and r3 generate two

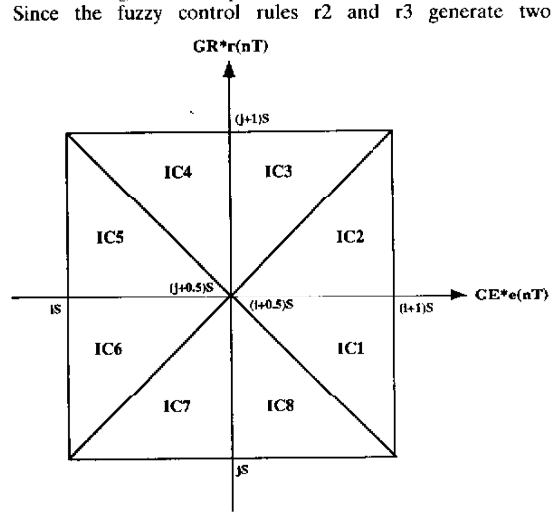

FIG. 2. Possible input combinations (IC) of scaled error, e^*, and scaled rate change of error, r^*, of process output which must be considered to carry out the Min operation in (r1*) to (r4*) when both e^* and r^* are within the interval $[-L, L]$.

TABLE 2. RESULTS OF EVALUATING THE MIN OPERATIONS IN (r1*) TO (r4*) FOR ALL COMBINATIONS OF INPUTS USING ZADEH FUZZY LOGIC AND (MIN) WHEN SCALED ERROR AND RATE CHANGE OF ERROR OF PROCESS OUTPUT ARE WITHIN THE INTERVAL $[-L, L]$. THE INPUT COMBINATIONS OF SCALED ERROR AND RATE CHANGE OF ERROR ARE SHOWN GRAPHICALLY IN FIG. 2

Region	r1*	r2*	r3*	r4*
IC1 and IC2	$\mu_{j+1}(r^*)$	$\mu_j(r^*)$	$\mu_i(e^*)$	$\mu_i(e^*)$
IC3 and IC4	$\mu_{i+1}(e^*)$	$\mu_j(r^*)$	$\mu_i(e^*)$	$\mu_j(r^*)$
IC5 and IC6	$\mu_{i+1}(e^*)$	$\mu_{i+1}(e^*)$	$\mu_{j+1}(r^*)$	$\mu_j(r^*)$
IC7 and IC8	$\mu_{j+1}(r^*)$	$\mu_{i+1}(e^*)$	$\mu_{j+1}(r^*)$	$\mu_i(e^*)$

memberships for the same member, $U_{-(i+j+1)}$, the equation (2.14) is needed to calculate the combined membership for $U_{-(i+j+1)}$. For the ICI to IC4 regions,

$$\mu_i(e^*) + \mu_j(r^*) = 1 - \frac{[e^* - (i+0.5)S] + [r^* - (j+0.5)S]}{S} \le 1, \quad (2.20)$$

because

$$0 \le [e^* - (i+0.5)S] + [r^* - (j+0.5)S] \le S. \quad (2.21)$$

Similarly, it is easy to prove

$$\mu_{i+1}(e^*) + \mu_{j+1}(r^*) \le 1, \text{ for IC5 to IC8 regions.} \quad (2.22)$$

Hence, the combined membership for $U_{-(i+j+1)}$ is always the sum of the memberships being ORed. Replacing the Min operations in (r1*) to (r4*) with their corresponding outcomes in Table 2 and using the defuzzification algorithm (2.15) in connection with (2.7), (2.9), (2.18) and (2.19), the scaled incremental output, $GU \cdot \Delta u(nT)$, for all eight regions can be found as follows:
for IC1, IC2, IC5 and IC6 regions

$$GU \cdot \Delta u(nT) = -(i+j+1)\frac{GU \cdot H}{N-1} - \frac{[GE \cdot e(nT) - (i+0.5)S] + [GR \cdot r(nT) - (j+0.5)S]}{2S - 2\,|GE \cdot e(nT) - (i+0.5)} \times \frac{GU \cdot H}{N-1}, \quad (2.23)$$

for IC3, IC4, IC7 and IC8 regions

$$GU \cdot \Delta u(nT) = -(i+j+1)\frac{GU \cdot H}{N-1} - \frac{[GE \cdot e(nT) - (i+0.5)S] + [GR \cdot r(nT) - (j+0.5)S]}{2S - 2\,|GR \cdot r(nT) - (j+0.5)S|} \times \frac{GU \cdot H}{N-1}. \quad (2.24)$$

$GU \cdot \Delta u(nT)$ consists of two parts. The first part is $(i+j+1)GU \cdot H/(N-1)$, which is a two-dimensional multilevel relay with respect to i and j. Note that the multilevel relay, denoted as Relay(i, j) can be rewritten as

$$\begin{aligned} \text{Relay}(i, j) &= -(i+j+1)\frac{GU \cdot H}{N-1} \\ &= -((i+0.5)S + (j+0.5)S)\frac{GU \cdot H}{S(N-1)}, \\ &= -((i+0.5)S + (j+0.5)S)\frac{GU \cdot H}{2L}. \quad (2.25) \end{aligned}$$

The point $((i+0.5)S, (j+0.5)S)$ is the coordinate of the center of the square shown in Fig. 2. Evidently, the multilevel relay contributes its control action according to the absolute position, with respect to the entire scaled input state plane, of the center of the square in which the current scaled input state (e^*, r^*) lies. Therefore, the author calls the multilevel relay a "global" multilevel relay. The second part of $GU \cdot \Delta u(nT)$ is a nonlinear nonfuzzy controller, which is denoted as $\delta u(i, j)$. The equations (2.23) and (2.24) indicate that $\delta u(i, j)$ is calculated according to the relative position of the current scaled input state $(GE \cdot e(nT), GR \cdot r(nT))$ with respect to the center of the square, $((i+0.5)S, (j+0.5)S)$, in which the current scaled input state lies. Therefore, one can see that the role of the nonlinear controller is to locally adjust the control action generated by the global multilevel relay. The author calls such a controller a "local" nonlinear controller.

A regular discrete-form linear PI controller whose output becomes zero when its inputs, $e(nT)$ and $r(nT)$, reach a steady-state $((i+0.5)S/GE, (j+0.5)S/GR)$, can be expressed as

$$\delta u_{PI}(i, j) = -\left(K_i\left[e(nT) - \frac{(i+0.5)S}{GE}\right] + K_p\left[r(nT) - \frac{(j+0.5)S}{GR}\right]\right), \quad (2.26)$$

where K_p and K_i are the proportional-gain and integral-gain, respectively. Therefore, the local nonlinear controller is actually a nonlinear PI controller with a local and changing steady-state $((i+0.5)S/GE, (j+0.5)S/GR)$:

$$\delta u(i, j) = -\left(K_i(e^*, r^*)\left[e(nT) - \frac{(i+0.5)S}{GE}\right] + K_p(e^*, r^*)\left[r(nT) - \frac{(j+0.5)S}{GR}\right]\right). \quad (2.27)$$

The proportional-gain and integral-gain change with input states and are described in equation (3.1).
(B) Either e^* or r^* is outside the interval $[-L, L]$.

To analytically describe the behavior of the nonlinear fuzzy controller when either e^* or r^* is outside the interval $[-L, L]$, the author divides the scaled input state plane outside the square configured by the interval $[-L, L]$ on the scaled error axis and the interval $[-L, L]$ on the scaled rate axis into 12 regions, as shown in Fig. 3. By using the same method described above, $GU \cdot \Delta u(nT)$ can be analytically derived for the regions, as shown in Table 3. According to Table 3, the nonlinear fuzzy controller becomes the sum of a global one-dimensional multilevel relay and a local linear P controller with a local and changing steady-state for the IC9, IC10, IC13 and IC14 regions, and the sum of a global one-dimensional multilevel relay and a local linear integral (I) controller with a local and changing steady-state for the IC11, IC12, IC15 and IC16 regions. The nonlinear fuzzy controller generates its maximum increment $(GU \cdot H)$ and decrement $(-GU \cdot H)$ in the IC19 and IC17 regions, respectively. For the IC18 and IC20 regions, the increment is zero. ■

It should be noted that when a scaled input state (e^*, r^*) is on a boundary of two adjacent regions, $GU \cdot \Delta u(nT)$ calculated by using the formula of either region is the same. In other words, there is no discrepancy in control action.

3. *Properties of the nonlinear fuzzy controller with linear control rules*

3.1. *Dynamic change of local nonlinear PI controller gains.* Comparing (2.23) and (2.24) with (2.26), one can see that the proportional-gain and integral-gain of the local

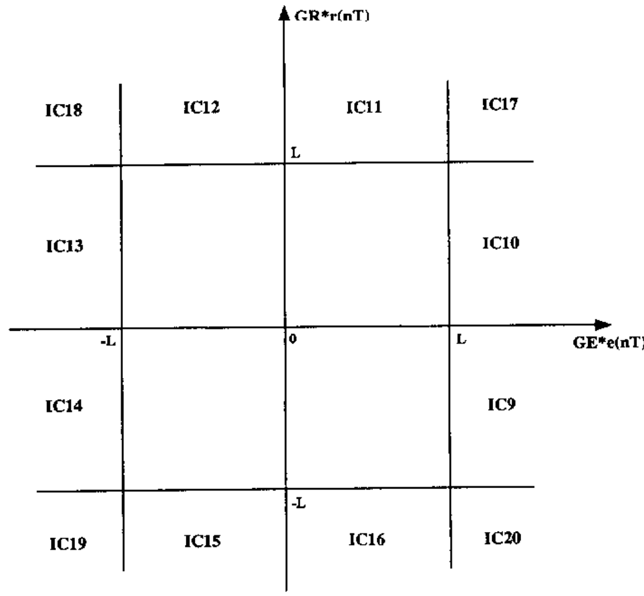

FIG. 3. Possible input combinations (IC) of scaled error, e^*, and scaled rate change of error, r^*, of process output which must be considered to carry out the Min operation in (r1*) to (r4*) when either e^* or r^* is outside the interval $[-L, L]$.

TABLE 3. THE SCALED INCREMENTAL OUTPUT OF THE FUZZY CONTROLLERS, $GU \cdot \Delta(nT)$, WHEN EITHER SCALED ERROR, $GE \cdot e(nT)$, OR SCALED RATE CHANGE OF ERROR, $GR \cdot r(nT)$, OF PROCESS OUTPUT IS OUTSIDE THE INTERVAL $[-L, L]$. THE INPUT COMBINATIONS OF SCALED ERROR AND RATE CHANGE OF ERROR ARE SHOWN GRAPHICALLY IN FIG. 2

IC9 and IC10	$-(J+j)\frac{GU \cdot H}{N-1} - (GR \cdot r(nT) - j \cdot S)\frac{GU \cdot H}{2L}$
IC11 and IC12	$-(i+J)\frac{GU \cdot H}{N-1} - (GE \cdot e(nT) - i \cdot S)\frac{GU \cdot H}{2L}$
IC13 and IC14	$-(-J+j)\frac{GU \cdot H}{N-1} - (GR \cdot r(nT) - j \cdot S)\frac{GU \cdot H}{2L}$
IC15 and IC16	$-(i-J)\frac{GU \cdot H}{N-1} - (GE \cdot e(nT) - i \cdot S)\frac{GU \cdot H}{2L}$
IC17	$-GU \cdot H$
IC18	0
IC19	$GU \cdot H$
IC20	0

nonlinear PI controller vary with input state and are

$$K_p(e^*, r^*) = \frac{GR \cdot GU \cdot H}{2(N-1)S}\beta(e^*, r^*) = \frac{GR \cdot GU \cdot H}{4L}\beta(e^*, r^*),$$
$$K_i(e^*, r^*) = \frac{GE \cdot GU \cdot H}{2(N-1)S}\beta(e^*, r^*) = \frac{GE \cdot GU \cdot H}{4L}\beta(e^*, r^*), \quad (3.1)$$

where

$$\beta(e^*, r^*) = \frac{S}{S - |GE \cdot e(nT) - (i+0.5)S|}, \quad \text{for IC1, IC2, IC5 and IC6 regions,}$$

and (3.2)

$$\beta(e^*, r^*) = \frac{S}{S - |GR \cdot r(nT) - (j+0.5)S|}, \quad \text{for IC3, IC4, IC7 and IC8 regions.}$$

Obviously, the local nonlinear PI controller can automatically adjust the proportional-gain and integral-gain to adapt to different scaled input states. The further the current scaled input state $(GE \cdot e(nT), GR \cdot r(nT)$ is from the center of the square $((i+0.5)S, (j+0.5)S)$ the larger the proportional-gain and integral-gain. With the constraints on e^* and r^* specified in (2.17), the range of the value of the nonlinear function $\beta(e^*, r^*)$ is calculated as

$$1 \le \beta(e^*, r^*) \le 2. \quad (3.3)$$

Hence, the ranges of the proportional-gain and integral-gain are

$$\frac{GR \cdot GU \cdot H}{4L} \le K_p(e^*, r^*) \le \frac{GR \cdot GU \cdot H}{2L},$$
$$\frac{GE \cdot GU \cdot H}{4L} \le K_i(e^*, r^*) \le \frac{GE \cdot GU \cdot H}{2L}. \quad (3.4)$$

3.2. *The role of the global multilevel relay and the local nonlinear PI controller in total control action and degree of nonlinearity.* The absolute value of maximum Relay(i, j) is $\text{Relay}_{max} = (N-2)GU \cdot H/(N-1)$, which is achieved when

$i = j = J - 1$ or $i = j = -J$. The absolute value of maximum $\delta u(i, j)$ is $\delta u_{max} = GU \cdot H/(N-1)$, which is achieved when $GE \cdot e(nT) = (i+1)S$ and $GR \cdot r(nT) = (j+1)S$ or when $GE \cdot e(nT) = i \cdot S$ and $GR \cdot r(nT) = j \cdot S$. The author defines the ratio

$$\rho = \frac{\delta u_{max}}{\text{Relay}_{max} + \delta u_{max}} \times 100\% = \frac{1}{N-1} \times 100\%, \quad (3.5)$$

to describe (1) the role of the local nonlinear PI controller and the role of the global multilevel relay in total control action; and (2) the degree of nonlinearity of the nonlinear fuzzy controllers as N changes. The smaller the ratio ρ, the less significant the role of the local nonlinear PI controller in total control action and the more significant the role of the global multilevel relay in total control action. When $N = 3$, ρ reaches its maximum, 50%, which indicates that the local nonlinear PI controller plays as important a role as does the global multilevel relay.

According to (3.4), the ranges of the proportional-gain and integral-gain are independent from N. That means that the ability of the local nonlinear PI controller to adapt locally to input state is the same for fuzzy controllers as N changes. However, it should be noted that the role of the local nonlinear PI controller in total control action is governed by the ratio ρ and therefore is different when N is different.

The ratio ρ also describes the degree of nonlinearity of the nonlinear fuzzy controllers. The smaller the ratio ρ, the finer the resolution of the output of the global multilevel relay and therefore the less nonlinear the fuzzy controller.

3.3. *The structure of the nonlinear fuzzy controllers when* $N \to \infty$.

Theorem 2 (*Limit theorem*). The nonlinear fuzzy controller with linear control rules becomes a linear PI controller as $N \to \infty$.

Proof. When $N \to \infty (J \to \infty)$, the control action from the local nonlinear PI controller approaches zero according to (2.23) and (2.24), that is $\delta u(i, j) \to 0$. On the other hand, the control action from the global multilevel relay becomes

$$\text{Relay}(i, j) = -(i+j+1)\frac{GU \cdot H}{2J} \to -\frac{GU \cdot H}{2}\left(\frac{i}{j} + \frac{j}{J}\right). \quad (3.6)$$

Because

$$iS = i\frac{L}{J}, \quad (i+1)S = (i+1)\frac{L}{J},$$
$$jS = j\frac{L}{J}, \quad (j+1)S = (j+1)\frac{L}{J}, \quad (3.7)$$

the inequalities (2.17) can be written as

$$\frac{i}{J} \le \frac{e^*}{L} \le \frac{i+1}{J} \quad \text{and} \quad \frac{j}{J} \le \frac{r^*}{L} \le \frac{j+1}{J}, \quad (3.8)$$

and hence

$$\frac{e^*}{L} \to \frac{i}{J} \quad \text{and} \quad \frac{r^*}{L} \to \frac{j}{L}, \quad (3.9)$$

when $N \to \infty$ (therefore $J \to \infty$, $i \to \infty$ and $j \to \infty$). Substituting (3.9) into (3.6), yield

$$\text{Relay}(i, j) = -\frac{GU \cdot H}{2L}(e^* + r^*)$$
$$= -\left(\frac{GU \cdot H \cdot GE}{2L} e(nT) + \frac{GU \cdot H \cdot GR}{2L} r(nT)\right), \quad (3.10)$$

and hence

$$GU \cdot \Delta u(nT) = \text{Relay}(i, j)$$
$$= -\left(\frac{GU \cdot H \cdot GE}{2L} e(nT) + \frac{GU \cdot H \cdot GR}{2L} r(nT)\right). \quad (3.11)$$

Therefore, one can immediately conclude that the nonlinear fuzzy controller (and the global multilevel relay) becomes a regular linear PI controller when N is ∞ (ρ is zero). The corresponding K_p and K_i are

$$K_p = \frac{GR \cdot GU \cdot H}{2L} \quad \text{and} \quad K_i = \frac{GE \cdot GU \cdot H}{2L}. \quad \blacksquare \quad (3.12)$$

4. *Relationship between fuzzy controllers with* $N \ge 3$ *and that with* $N = 2$

In Ying *et al.* (1990), we analytically proved that a simplest possible ($N = 2$) nonlinear fuzzy controller constructed in the same way as those in this paper was a nonlinear PI controller:

for IC1, IC2, IC5 and IC6 regions

$$du(NT) = -\frac{GE \cdot e(nT) + GR \cdot r(nT)}{2L - GE \cdot |e(nT)|}\frac{GU \cdot H}{2}, \quad (4.1)$$

or for IC3, IC4, IC7 and IC8 regions

$$du(nT) = -\frac{GE \cdot e(nT) + GR \cdot r(nT)}{2L - GR \cdot |r(nT)|}\frac{GU \cdot H}{2}, \quad (4.2)$$

where

$$GE \cdot |e(nT)| \le L \quad \text{and} \quad GR \cdot |r(nT)| \le L. \quad (4.3)$$

Based on (4.1) and (4.2), the simplest possible nonlinear fuzzy controller does not include the multilevel relay and hence the nonlinear PI controller is a global controller having one single global and fixed steady-state (0, 0). It can be easily proven that the fuzzy controller with $N = 2$ can be expressed as

$$du(nT) = -[K_i(e^*, r^*)e(nT) + K_p(e^*, r^*)r(nT)], \quad (4.4)$$

where the gains and the ranges of the gains are the same as those in (3.1) and (3.4), which indicates that the local nonlinear PI controller can adjust its gains to adapt to different input states as much as the global nonlinear PI controller can. However, there is a fundamental difference between these two nonlinear PI controllers. That is, the role the local nonlinear PI controller can play in total control action is less significant because the role is governed by the ratio ρ. The role is small when N is large. On the other hand, the global nonlinear PI controller contributes sole control action and therefore its ratio ρ is 100% by definition, which also means the fuzzy controller with $N = 2$ is more nonlinear than any other fuzzy controllers.

5. *Conclusions*

The author concludes that the nonlinear fuzzy controllers with $N \ge 3$ consist of a global multilevel relay and a local nonlinear PI controller similar to the global nonlinear PI controller when $N = 2$ but with a local and changing steady-state $((i + 0.5)S/GE, (j + 0.5)S/GR)$. The consequences of employing more than two members ($N \ge 3$) for input fuzzy sets are (1) introducing the global multilevel relay with resolution $GU \cdot H/(N-1)$; and (2) reducing the role of the local nonlinear PI controller in total control action from 100% to $1/(N-1)$. Larger N makes the fuzzy controllers less nonlinear. As N approaches ∞, the nonlinearity disappears and the fuzzy controller becomes a linear PI controller. The degree of nonlinearity of the fuzzy controllers and the role of the local nonlinear PI controllers in total control action are quantitatively described by introducing the ratio ρ. The fuzzy controller with $N = 2$, whose ρ is 100%, is the most nonlinear fuzzy controller.

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References

Buckley, J. J. (1989a). Further results for the linear fuzzy controller. *Kybernetes*, **18**, 48–55.

Buckley, J. J. (1989b). Nonlinear fuzzy controller. *Information Sciences*, to appear.

Buckley, J. J. and H. Ying (1989). Fuzzy controller theory:

limit theorems for linear fuzzy control rules. *Automatica*, **25**, 469–472.
Buckley J. J. and H. Ying (1990). Linear fuzzy controller: it is a linear nonfuzzy controller. *Information Sciences*, **51**, 183–192.
Buckley, J. J. (1990). Further limit theorems for linear control rules. *Fuzzy Sets and Systems*, **36**, 225–233.
Siler, W. and H. Ying (1989). Fuzzy control theory: the linear case. *Fuzzy Sets and Systems*, **33**, 275–290.
Wang, P. Z., H. M. Zhang and W. Xu (1990). Pad-analysis of fuzzy control stability. *Fuzzy Sets and Systems*, **38**, 27–42.
Ying, H. (1987). Fuzzy control theory. Technical Report. Department of Biomedical Engineering, University of Alabama at Birmingham.
Ying, H., W. Siler and J. J. Buckley (1988). Fuzzy control theory: a nonlinear case. *Proc. of NASA's Conference on Artificial Neural Systems and Fuzzy Logic*, Houston, U.S.A., 2–3 May.
Ying, H., W. Siler and J. J. Buckley (1990). Fuzzy control theory: a nonlinear case. *Automatica*, **26**, 513–520.