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Brief Paper

An Analytical Study on Structure, Stability and Design of General Nonlinear Takagi–Sugeno Fuzzy Control Systems*

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Key Words—Fuzzy control; fuzzy modeling; fuzzy systems; PID controllers; stability; variable structure controllers; variable gain controllers; Takagi–Sugeno fuzzy control.

Abstract—In this paper, we first study analytical structure of general nonlinear Takagi-Sugeno (TS, for short) fuzzy controllers, then establish a condition for analytically determining asymptotic stability of the fuzzy control systems at the equilibrium point, and finally use the stability condition in design of the control systems that are at least locally stable. The general TS fuzzy controllers use arbitrary input fuzzy sets, any types of fuzzy logic AND, TS fuzzy rules with linear consequent and the generalized defuzzifier which contains the popular centroid defuzzifier as a special case. We have mathematically proved that the general TS fuzzy controllers are nonlinear controllers with variable gains continuously changing with controllers' input variables. Using Lyapunov's linearization method, we have established a necessary and sufficient condition for analytically determining local asymptotic stability of TS fuzzy control systems, each of which is made up of a fuzzy controller of the general class and a nonlinear plant. We show that the condition can be used in practice even when the plant model is not explicitly known. We have utilized the stability condition to design, with or without plant model, general TS fuzzy control systems that are at least locally stable. Three numerical examples are given to illustrate in detail how to use our new results. Our results offer four important practical advantages: (1) our stability condition, being a necessary and sufficient one, is the tightest possible stability condition, (2) the condition is simple and easy to use partially because it only needs the fuzzy controller structure around the equilibrium point, (3) the condition can be used for determining system local stability and designing fuzzy control systems that are stable at least around the equilibrium point even when the explicit plant models are unavailable, and (4) the condition covers a very broad range of nonlinear TS fuzzy control systems, for which a meaningful global stability condition seems impossible to establish. © 1998 Elsevier Science Ltd. All rights reserved.

1. *Introduction*

There are two major types of fuzzy controllers: Mamdani (1974) fuzzy controllers Wang (1994) and Takagi-Sugeno (1985) (TS, for short) fuzzy controllers. The major difference is that Mamdani fuzzy controllers use fuzzy sets whereas TS fuzzy controllers employ (linear) functions of input variables in the consequent of fuzzy control rules. Significant efforts have been made to reveal and understand Mamdani fuzzy controllers in relation to conventional controllers, such as PID (proportional-integral-derivative) control and variable gain control (e.g., Hajjaji and Rachid, 1994; Langari, 1992; Lewis and Liu, 1996; Matia *et al.*, 1992; Wang *et al.*, 1990; Wang *et al.*, 1993; Xu *et al.*, 1996a, b; Ying *et al.*, 1990; Ying, 1993a, b; Ying, 1994b). Since stability is always one of the central issues for any control systems, stability of the systems involving Mamdani fuzzy controllers have been investigated (e.g., Farinwata and Vachtsevanos, 1993; Furutani *et al.*, 1992; Chen and Ying, 1998; Chen *et al.*, 1995; Kang, 1993; Kang and Vachtsevanos, 1993; Kania *et al.*, 1980; Kiszka *et al.*, 1985; Langari and Tomizuka 1990a, b; Langari and Tomizuka, 1993; Malki *et al.*, 1994; Ying, 1993a, 1994a). TS fuzzy controllers were developed in 1985 and quickly gained popularity, especially in the past few years. However, TS fuzzy controllers, just like Mamdani fuzzy controllers, have been treated and used in most cases as black-box controllers. Little is known about possible relationship between TS fuzzy controllers and conventional controllers. The only results available at present are those obtained recently by present author (Ying, 1988a, b). Stability results on TS fuzzy control systems are scarce. The results in existence are some sufficient global stability conditions for a class of TS fuzzy controllers controlling TS fuzzy models (e.g., Tanaka and Sugeno, 1990, 1992; Tanaka and Sano, 1993, 1994; Tanaka *et al.*, 1996; Wang *et al.*, 1996). The conditions were also utilized to design robust TS fuzzy control systems (Tanaka *et al.*, 1996).

The objectives of our research presented in this paper were: (1) to reveal analytical structure of general nonlinear TS fuzzy controllers relative to conventional controllers, specifically to variable gain controllers, (2) to develop a simple and practical stability condition for determining local stability of the general TS fuzzy control systems involving nonlinear plants whose mathematical models may or may not be explicitly available, and (3) to use the stability condition for designing general nonlinear TS fuzzy control systems that are at least locally stable under the assumption that the mathematical models of the plants may or may not be available. The general nonlinear TS fuzzy controllers in our study use arbitrary input fuzzy sets, any types of fuzzy logic AND operators, TS fuzzy control rules with linear consequent and the generalized defuzzifier which contains the popular centroid defuzzifier as a special case.

To be more specific, we formulate the structure, stability and design problems addressed in this paper as follows. Suppose that we have a nonlinear plant and want to control it by a TS fuzzy controller of the general class. We denote the plant model as $P(\mathbf{x})$, which may or may not be mathematically available. Without loss of generality, we assume, throughout this paper, that (1) $\mathbf{x} = 0$ is the equilibrium point of the nonlinear TS fuzzy control system consisting of the plant and a TS fuzzy controller of the general class, and (2) $P(\mathbf{x})$ is linearizable about the

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equilibrium point. To be practical, we assume that only the fuzzy controller structure around the equilibrium point is obtainable and that the controller structure for the whole input space cannot be derived analytically, which is a reasonable assumption for the majority of the fuzzy controllers. Based on these assumptions, we answer the following five important questions:

(1) What is the analytical structure of the general TS fuzzy controllers relative to conventional controllers?
(2) when $P(\mathbf{x})$ is available, how to analytically determine asymptotic stability of the TS fuzzy control system at the equilibrium point?
(3) When $P(\mathbf{x})$ is mathematically unavailable but is known linearizable about the equilibrium point, how to analytically determine local stability of the nonlinear TS fuzzy control system?
(4) When $P(\mathbf{x})$ is available, how to analytically design a TS fuzzy controller of the general class to achieve asymptotic system stability at least at the equilibrium point?
(5) When $P(\mathbf{x})$ is mathematically unavailable but is known linearizable about the equilibrium point, how to design a TS fuzzy controller of the general class that achieves asymptotic system stability at least at the equilibrium point?

The first question is about the relationship between the general nonlinear TS fuzzy controllers and conventional controllers. The second and third questions address local stability of the general nonlinear TS fuzzy control systems whereas the last two questions are on the design of general nonlinear TS fuzzy control systems that are stable at least around the equilibrium point.

The reader may wonder why we wanted to focus on local stability instead of global stability. Our motivation and justification are explained as follows. In our opinion, both types of system stability are equally important, and one type cannot replace the other as each type has its distinctive advantages and disadvantages. A global stability condition determines system stability in the whole input space. For nonlinear systems, global stability conditions are, in most cases, sufficient conditions, and necessary ones are uncommon. Except for linear systems, it is rare that a global stability condition is a necessary and sufficient condition. The most widely used and effective methodology for global stability determination is the one developed by Lyapunov, which requires a Lyapunov function to be found for the (fuzzy) control system involved. Regardless of methodologies, their foremost assumption/requirement for establishing a global stability condition, sufficient or necessary, is that the analytical expressions of both the (nonlinear) controller and the (nonlinear) plant model are explicitly available. This assumption/requirement is critical: global stability analysis is impossible if it is not satisfied. For the Lyapunov methods, for instance, without knowing the complete structure of the fuzzy controller and the model of the plant, a correct Lyapunov function is impossible to construct for the fuzzy control system.

This assumption/requirement, however, is rather unrealistic and impractical not only to the general nonlinear TS fuzzy controllers in this paper, but also to fuzzy control as a whole. First of all, explicit structures of many fuzzy controllers are not analytically derivable. A fuzzy controller is made up of several inter-related nonlinear components: input fuzzy sets, output fuzzy sets, fuzzy rules, fuzzy logic operators and a defuzzifier. The structures and parameters of these components are chosen by the controller designer at will and are with little restriction. As such, the structures of most fuzzy controllers are inherently complex and can virtually be any nonlinear forms, making analytical derivation of the complete structures for the whole input space virtually impossible. Second, the plant model involved, which is assumed to be nonlinear in this paper, may or may not be available explicitly. A nonlinear model may be developed by two approaches. The first approach analytically derives a plant model by using the natural laws governing the plant. This approach often fails if the plant is too complicated. The second approach is system identification using data representing the plant's behavior, which is a black-box approach. Accurate nonlinear system identification is known to be difficult because very often the modeler cannot even correctly assume a nonlinear structure for the model to begin with.

We should further point out that even if the assumption/requirement is met, properly determining global stability for general nonlinear TS fuzzy control systems is still very difficult or likely impossible. Lyapunov function construction currently is more an art than science and heavily involves trial and error. Due to the structural and parametric complexity of these general fuzzy controllers, finding a proper Lyapunov function for all of the possible nonlinear fuzzy control systems, which are infinite in number, is virtually impossible.

Given these difficulties associated with global stability determination, we decided to concentrate on local stability. Determining local stability of the general TS fuzzy control systems requires much less information and assumption on fuzzy controllers and plant models. For our work in this paper, we only need to know: (1) the fuzzy controller structure around the equilibrium point, and (2) the linearizability of the plant model at the equilibrium point, both of which are obtainable in many cases. Thus, the results presented in this paper are practically usable.

We should also point out that local stability does not mean system stability at the equilibrium point only. Rather, it means system stability in a region around the equilibrium point. Depending on the system, the region can be large enough to cover the part of the input space of interest.

In the next section, we define the components of the general TS fuzzy controllers.

2. Configuration of general TS fuzzy controllers

The general TS fuzzy controllers employ M discrete-time input variables: namely $x_1(n), x_2(n), \ldots, x_M(n)$, where n is sampling time. Input variable $x_i(n)$ is fuzzified by P_i ($P_i > 1$) arbitrary-shape input fuzzy sets. We denote the jth input fuzzy set for $x_i(n)$ as A_{ij} and denote its membership function as $\mu_{ij}(x_i)$, where $j = 1, 2, \ldots, P_i$. For M input variables, $P_1 \times P_2 \times \cdots \times P_M$ different combinations of the input fuzzy sets exist and that many fuzzy control rules are needed to cover all the combinations. We use Ω to represent the total number of the fuzzy control rules:

$$\Omega = P_1 \times P_2 \times \cdots \times P_M = \prod_{i=1}^{M} P_i.$$

The TS fuzzy control rules with linear rule consequent (Takagi and Sugeno, 1985) are used and the jth rule of our general TS fuzzy controllers is ($1 \le j \le \Omega$):

$$\text{IF } x_1(n) \text{ is } A_{1j} \text{ AND...AND } x_M(n) \text{ is } A_{Mj}$$

$$\text{THEN } u_j(n) = a_{0j} + a_{1j}x_1(n) + \cdots + a_{Mj}x_M(n), \qquad (1)$$

where a_{0j} and a_{ij}'s are adjustable design parameters. To combine the M membership values of the input fuzzy sets in the rule antecedent, any types of fuzzy logic AND operators may be used and different types of AND operators may be used in different rules. Using $\otimes$ as a symbol for arbitrary types of fuzzy logic AND operators, the combined membership for consequent $u_j(n)$ in the jth rule is

$$\mu_j(\mathbf{x}) = \mu_{1j}(x_1) \otimes \mu_{2j}(x_2) \otimes \ldots \otimes \mu_{Mj}(x_M),$$

where

$$\mathbf{x} = (x_1(n), x_2(n), \ldots, x_M(n)).$$

When $x_i(n) = 0$ for all i, $\mathbf{x} = 0$ and $\mu_j(\mathbf{x})$ at that time is denoted as $\mu_j(0)$.

The general TS fuzzy controllers use the generalized defuzzifier (Yager and Filev, 1994) to combine $u_j(n)$'s, $j = 1, 2, \ldots \Omega$. After defuzzification, the output of the general fuzzy controllers is:

$$u(n) = \frac{\sum_{j=1}^{\Omega} \mu_j^{\alpha}(\mathbf{x}) \cdot u_j(n)}{\sum_{j=1}^{\Omega} \mu_j^{\alpha}(\mathbf{x})} = \frac{\sum_{j=1}^{\Omega} \mu_j^{\alpha}(\mathbf{x})(a_{0j} + a_{1j}x_1(n) + \ldots + a_{Mj}x_M(n))}{\sum_{j=1}^{\Omega} \mu_j^{\alpha}(\mathbf{x})}. \qquad (2)$$

Different defuzzification results can be obtained by using different α value (Yager and Filer, 1994), where $0 \le \alpha < \infty$. The popular centroid defuzzifier and mean of maximum defuzzifier are just two special cases when $\alpha = 1$ and ∞, respectively.

3. *Results on structure, stability and design of the general TS fuzzy control systems*

In conventional control theory, a discrete-time general linear controller with M input variable is

$$u(n) = c_0 + c_1x_1(n) + \cdots + c_Mx_M(n),$$

where c_i is constant gain for input variable $x_i(n)$ and c_0 is a constant control offset. In most cases, the offset is not needed and hence, without loss of generality, we use

$$u(n) = c_1x_1(n) + \cdots + c_Mx_M(n) \tag{3}$$

to represent a general linear controller in this paper. Since PID control plays such an important role in industrial control, it is worth pointing out that in equation (3), if only $x_1(n)$, $x_2(n)$ and $x_3(n)$, defined as follows, are used:

$$x_1(n) = SP(n) - y(n), \quad x_2(n) = \sum_{i=1}^{n} x_1(i)\Delta t,$$

$$x_3(n) = \frac{x_1(n) - x_1(n-1)}{\Delta t},$$

where SP(n) is setpoint/reference signal for plant output and Δt is sampling period for the controller, then equation (3) becomes a linear PID controller with c_1, c_2 and c_3 being the constant proportional-gain, integral-gain and derivative-gain, respectively. Note that $x_1(n)$ is error of plant output.

Now, we rewrite equation (2) to represent the general TS fuzzy controllers in the following manner:

$$u(n) = b_0(\mathbf{x}) + b_1(\mathbf{x})x_1(n) + \cdots + b_M(\mathbf{x})x_M(n) \tag{4}$$

where

$$b_i(\mathbf{x}) = \frac{\sum_{j=1}^{\Omega} \mu_j^{\alpha}(\mathbf{x}) \cdot a_{ij}}{\sum_{j=1}^{\Omega} \mu_j^{\alpha}(\mathbf{x})} \quad \text{for } i = 0,1,\ldots,M. \tag{5}$$

Without loss of generality, we assume $a_{0j} = 0$ for all j in the TS fuzzy rules in equation (1). This makes equation (4) become

$$u(n) = b_1(\mathbf{x})x_1(n) + \cdots + b_M(\mathbf{x})x_M(n) \tag{6}$$

Comparing equation (6) with equation (3), one sees that the general TS fuzzy controllers are actually nonlinear controllers with variable gains changing with state of the input variables. This is so because $b_i(\mathbf{x})$, the gain for $x_i(n)$, is a nonlinear function of $\mathbf{x}$. Stating this result formally, we have:

Theorem 1. The general TS fuzzy controllers are nonlinear controllers with variable gains changing with state of input variables.

Having answered the first question that we raised in Introduction, we are going to answer the second and third questions concerning asymptotic stability around the equilibrium point. Recall that we have assumed earlier that the plant to be controlled is $P(\mathbf{x})$ which is linearizable about the equilibrium point. According to Lyapunov's linearization method (Slotine and Li 1991), a nonlinear TS fuzzy control system consisting of $P(\mathbf{x})$ controlled by a TS fuzzy controller of the general class is asymptotically stable around $\mathbf{x} = 0$ if and only if the linear control system consisting of the same $P(\mathbf{x})$ controlled by the linear controller

$$u(n) = b_1(0)x_1(n) + \cdots + b_M(0)x_M(n) \tag{7}$$

is asymptotically stable around $\mathbf{x} = 0$. The reader is reminded that, at $\mathbf{x} = 0$, the nonlinear TS fuzzy controller becomes the linear controller with $b_i(0)$ being constant gain for $x_i(n)$. We formally state this necessary and sufficient stability condition below.

Theorem 2. For any nonlinear TS fuzzy controller of the general class that is linearizable about the equilibrium point $\mathbf{x} = 0$, the TS fuzzy control system consisting of $P(\mathbf{x})$ controlled by the TS fuzzy controller is asymptotically stable if and only if the linearized TS fuzzy control system is asymptotically stable around the equilibrium point.

In equation (7), $b_i(0)$, for all i, can be calculated using equation (5) because a_{ij}'s and α are known for a given fuzzy controller and μ_j's and Ω are readily computable based on the membership functions of input fuzzy sets, fuzzy rules and fuzzy logic AND operators used in the fuzzy controller (the first numerical example in Section 4 shows this point in detail). We emphasize, however, that before the computed $b_i(0)$'s are used for the stability determination, one must first check whether the fuzzy control system is linearizable about the equilibrium point, as required by Lyapunov linearization method. The linearizability requires that the system be differentiable at least once with respect to all input variables and the resulting derivatives at the equilibrium point are unique for every input variable.

One may check the linearizability of the fuzzy controller and the plant to be controlled separately. If both of them are linearizable, the fuzzy control system is linearizable. Otherwise, the system is not linearizable. The linearizability of a TS fuzzy controller depends on the controller configuration (i.e. input fuzzy sets, fuzzy logic AND operators, fuzzy rules and the defuzzifier employed). When a TS fuzzy controller of the general class is configured by some widely used components (e.g. triangular, trapezoidal or Gaussian membership functions for input fuzzy sets, product fuzzy logic AND operator, the centroid defuzzifier, etc.), it is likely linearizable about the equilibrium point. Regardless of the components, one may be able to determine the linearizability of some fuzzy controllers even without explicit expressions of their structures. For example, if we know that structure of a fuzzy controller is a polynomial around the equilibrium point, then we know the controller is linearizable even without explicit knowledge about the polynomial (e.g. the degree of the polynomial and its coefficients). If such qualitative knowledge is not available, then analytical expression of the controller structure **around the equilibrium point** is necessary for the linearizability test. We emphasize that analytical expression of the complete controller structure for the whole input space is not needed. We will show these two important points more using an example TS fuzzy controller in Section 4 (i.e. Example 1).

Caution should be exercised in the linearizability test when a fuzzy controller uses Zadeh fuzzy logic AND operator (i.e. the minimum operation) in fuzzy control rules. This is because, when it is used, input space around the equilibrium point must be divided into several smaller regions in such a manner that in each region, the membership for one of the input variables is always less than or equal to those for the rest of the input variables. Subsequently, only one analytical expression describing structure of the fuzzy controllers is obtained in each region. The resulting structure may be different in different regions, which could potentially yield more than one derivative result at the equilibrium point, failing the linearizability test. Importantly, through, we have shown some fuzzy controllers using Zadeh fuzzy logic AND are linearizable about the equilibrium point (see Ying *et al.* (1990) for an example).

If a TS fuzzy control system is not linearizable about the equilibrium point, Theorem 2 should not be used to derive stability information. This, however, does not imply instability of the fuzzy control system. It simply means that the particular fuzzy control system fails to satisfy the assumption made in Theorem 2, making the theorem inapplicable to this particular fuzzy control system. Whenever this happens, one needs to use other stability analysis methods.

Theorem 2 can be used to determine stability not only when the plant model is explicitly available, but also when the plant model is unavailable but is known to be linearizable around the equilibrium point (indeed, most physical systems are). When this is the case, one can construct a nonlinear TS fuzzy controller of the general class that is linearizable about the equilibrium point and computes $b_i(0)$'s using equation (5). One then devises a linear controller whose input variables are the same as those of the TS fuzzy controller and whose gains are the computed $b_i(0)$'s. Now, use the linear controller to control the plant. If the resulting linear control system is observed to be locally stable (unstable), the fuzzy control system will be locally stable (unstable) too. The third example in Section 4 shows how to do this in more detail. Overall, the underlying idea here is similar to what we developed for the design of stable Mamdani fuzzy controllers (Ying, 1994a). This approach is of practical importance as in many real-world control problems, plants are often too complex and/or costly to be precisely modeled. Since any model is merely an approximation to the physical plant, it is rational to seek determination of system stability without the accurate plant model.

Local stability means system stability at least at the equilibrium point. Because of the system continuity, a region around the equilibrium point is also stable. The size and shape of the stable region (known as "basin") can be computed but, in most cases, the computation is rather involved and the result can be (very) conservative. It is possible that the basin is large enough to cover the part of the input space of interest. For practical applications, we argue that, because local stability guaranteed by Theorem 2 provides a good starting point, it would not be too difficult for one to tune the fuzzy control system manually using the trial and error method in achieving global stability. After all, many real-world PID control systems are tuned manually to obtain satisfactory performance as well as global stability.

Theorem 2 offers four practically important advantages. First of all, it is a necessary and sufficient condition. Thus, unlike the sufficient or necessary conditions for global stability in the literature, which often are (very) conservative, our condition is the tightest possible stability condition. Second, minimal information on the fuzzy controller and plant is required. Only the controller structure around the equilibrium point is necessary and the plant model is not required if it is known to be linearizable at the equilibrium point. Finally, Theorem 2 covers a very broad class of TS fuzzy control systems. None of these merits would be possible if global stability, instead of local stability, was pursued.

Theorem 2 can not only be used to determine local stability but also to design the general TS fuzzy control systems that are at least stable around the equilibrium point. This leads to the last two questions asked in Introduction regarding the design of the fuzzy control systems with or without plant models. The technique that we have developed using Theorem 2 is as follows.

We first design and run a linear controller (3) which can control, with asymptotic stability around the equilibrium point, the linearized plant model when $P(\mathbf{x})$ is mathematically available or control the physical plant itself when $P(\mathbf{x})$ is mathematically unavailable. We denote the gain for $x_i(n)$ as d_i. Hence, the linear controller that achieves local stability is:

$$u(n) = d_1 x_1(n) + \cdots + d_M x_M(n).$$

To design a TS fuzzy controller of the general class that makes the fuzzy control system asymptotically stable around the equilibrium point, we let $b_i(0) = d_i$ for all i. In other words, we let

$$\frac{\sum_{j=1}^{\Omega} \mu_j^{\alpha}(0) \cdot a_{ij}}{\sum_{j=1}^{\Omega} \mu_j^{\alpha}(0)} = d_i, \qquad i = 1,2,\ldots,M, \tag{8}$$

and obtain M equations. Obviously, one solution set is $a_{ij} = d_i$ for all i and j. But this solution set makes the general TS fuzzy controllers in equation (6) a linear controller. This is because when $a_{ij} = d_i$, all the rule consequent in Ω control rules become the same:

$$u_j(n) = d_1 x_1(n) + \cdots + d_M x_M(n), \quad j = 1,2,\ldots,\Omega,$$

which means that, regardless of the input fuzzy sets used and input state (i.e. any combination of the input variables), the output of the general TS fuzzy controllers is always the same as that of the linear controller. Therefore, the solution set $a_{ij} = d_i$ is not an appropriate one for our purpose.

To find other solution sets that make the fuzzy controller both locally stable and nonlinear, we solve equation (8) for the unknowns, namely $\mu_j^{\alpha}(0)$'s and a_{ij}'s for all i and j. Note that $\mu_j^{\alpha}(0)$'s are not really unknowns because they may be determined first to reduce the number of unknowns. In the design of a fuzzy controller, the designer needs to use his intuition and/or insightful knowledge to properly construct the membership functions, select a proper defuzzifier and choose appropriate fuzzy logic AND operators. Numerous successful applications of fuzzy control have indicated that: (1) triangular, trapezoidal and Gaussian membership functions are excellent choices for the input fuzzy sets, and (2) the centroid defuzzifier (i.e. $\alpha = 1$) works very well. As for the fuzzy logic, there are only two types that have been widely used: product AND operator and Zadeh AND operator. We should point out that this part of design process cannot be fully automated and human judgement must be involved because (1) these components determine the controller structure and hence are nonparametric; and (2) there are too many degree of freedom for the nonparametric choices. The only imaginable way to "automate" this part of the design process seems to involve global optimization schemes, such as genetic algorithms, for blindly optimizing the components. Such an approach would be time consuming and would involve great trial-and-error efforts. It is not clear whether such an approach should be considered as a design approach in the sense of traditional control theory.

Once these three controller components are decided, $\mu_j^{\alpha}(0)$'s can be computed and we will then only need to solve equation (8) for a_{ij}'s. This involves solving a set of M linear equations that are independent each other. Each equation typically contains more than one but up to Ω unknowns. Obviously, there exist an infinite number of solution sets but they can be found rather easily. One way is as follows.

In equation (8), we let,

$$a_{i1} = \beta_{i1}, \quad a_{i2} = \beta_{i2}, \ldots, \quad a_{i\Omega-1} = \beta_{i\Omega-1}, \quad \text{for all } i$$

where $\beta_{i1}, \beta_{i2}, \ldots, \beta_{i\Omega-1}$ are constants that we choose at will. We then compute $a_{i\Omega}$:

$$a_{i\Omega} = \frac{\sum_{j=1}^{\Omega-1} \mu_j^{\alpha}(0)(d_i - \beta_{ij}) + \mu_{\Omega}^{\alpha}(0)}{\mu_{\Omega}^{\alpha}(0)}, \quad \text{for all } i. \tag{9}$$

This will produce one solution set.

We point out that a_{ij}'s such computed satisfy equation (8) and thus guarantees asymptotic stability of the fuzzy control system around the equilibrium point. However, performance of the fuzzy control system is not guaranteed, which is the case for any method that solves for a_{ij}'s based on equation (8) alone. This, however, should not be regarded as a problem because most nonlinear control systems designed by using conventional control theory also only guarantee system stability. We suggest that one first computes many solution sets and then chooses, using computer simulation combined with trial-and-error effort, one solution set that enables the fuzzy control system to perform satisfactorily. We stress that this non-unique solution problem is a universal problem associated with any TS fuzzy controllers because, for one thing, there are so many adjustable system parameters in the fuzzy controllers.

Global stability of the TS fuzzy control systems is not guaranteed too when a_{ij}'s are calculated based on equation (8). Nevertheless, in light of the above discussion concerning Theorem 2, this should not be regarded as weakness of our design technique. After all, it is practically impossible to design such a general class of TS fuzzy control systems with global stability anyway. It is conceivable, however, that if one properly divides the general fuzzy control systems into a number of groups of less general fuzzy control systems, one may be able to establish global stability conditions for some or all of the systems groups (e.g., Langari and Tomizuka (1990)).

4. *Numerical examples for illustrating the stability determination and design of locally stable TS fuzzy control systems*

In this section, we use three examples, which are related to each other, to demonstrate how to utilize our new results to determine asymptotic stability of a TS fuzzy control system around the equilibrium point and how to design a locally stable TS fuzzy control system. To make our presentation clearer, we assume that plant model is known in the first two example and assume the opposite in the last example. We use a forced pendulum as our plant whose motion is described by a second-order nonlinear differential equation (Franklin *et al.*, 1992):

$$\ddot{\theta} + b\dot{\theta} + \omega^2 \sin\theta = \frac{T_{\text{in}}(t)}{mL^2}. \tag{10}$$

where θ is the angle between a rigid rod holding the pendulum and a vertical line, b is the coefficient of viscous friction, m is the mass of the pendulum, L is the length of the rod, $\omega^2 = g/L$ and

$T_{in}(t)$ is input torque applied to the rod. After linearized about the equilibrium point $\theta = 0$, (10) becomes

$$\ddot{\theta} + b\dot{\theta} + \omega^2\theta = \frac{T_{in}(t)}{mL^2}. \tag{11}$$

We use $mL^2 = 1$, $b = 0.25$, $\omega^2 = 100$ in our examples. To discretize the continuous-time model, we use a sampling period of 0.01 s (i.e. $\Delta t = 0.01$ s).

Example 1. Suppose that one wants to use a nonlinear TS fuzzy controller of the general class to control the forced pendulum. Assume that the controller has two input variables:

$$x_1(n) = \mathrm{SP}(n) - \theta(n) \text{ and } x_2(n) = (x_1(n) - x_1(n-1))/\Delta t$$

where SP(n) is setpoint/reference signal to the fuzzy controller. Note that $x_1(n)$ is error of plant output whereas $x_2(n)$ is rate change of error of plant output. The membership functions for $x_1(n)$ and $x_2(n)$ that one chooses are shown in Fig. 1, where $d_{11,a} = 1$, $d_{11,b} = 0.8$, $d_{12,a} = -0.7$, $d_{12,b} = 0.4$, $d_{21,a} = -1.1$, $d_{21,b} = 1.7$, $d_{22,a} = -0.6$ and $d_{22,b} = 2.1$. There are four control rules that cover the input space around the equilibrium point:

IF $x_1(n)$ is A_{11} AND $x_2(n)$ is A_{21} THEN $u_1(n)$
$= a_{11}x_1(n) + a_{21}x_2(n)$

IF $x_1(n)$ is A_{11} AND $x_2(n)$ is A_{22} THEN $u_2(n)$
$= a_{12}x_1(n) + a_{22}x_2(n)$

IF $x_1(n)$ is A_{12} AND $x_2(n)$ is A_{21} THEN $u_3(n)$
$= a_{13}x_1(n) + a_{23}x_2(n)$

IF $x_1(n)$ is A_{12} AND $x_2(n)$ is A_{22} THEN $u_4(n)$
$= a_{14}x_1(n) + a_{24}x_2(n)$,

where $a_{11} = 20$, $a_{21} = 1$, $a_{12} = 24$, $a_{22} = -18$, $a_{13} = 7$, $a_{23} = 2$, $a_{14} = 13$ and $a_{24} = 5$. Product fuzzy logic AND operator is used in the rules and the centroid defuzzifier is employed for defuzzification (i.e., $\alpha = 1$). The controller designer wants to know whether such designed TS fuzzy control system is asymptotically stable around the equilibrium point.

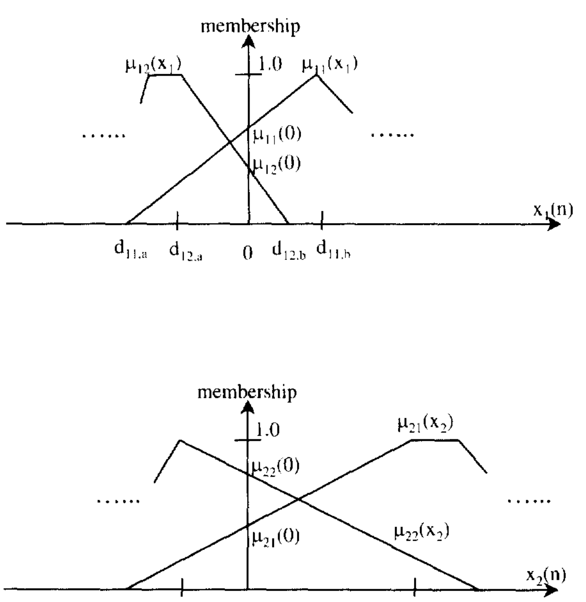

Fig. 1. The membership functions used to fuzzify $x_1(n)$ and $x_2(n)$ in the three examples.

Solution. The plant is linearizable and has been linearized above. We now need to determine the linearizability of the fuzzy controller that has been designed. According to (4), the fuzzy controller is described by

$$u(n) = b_1(\mathbf{x})x_1(n) + b_2(\mathbf{x})x_2(n) \tag{12}$$

where, using (5) and noting $\Omega = 4$ (only four rules are involved around the equilibrium point),

$$b_i(\mathbf{x}) = \frac{\sum_{j=1}^{4} \mu_j(\mathbf{x})a_{ij}}{\sum_{j=1}^{4} \mu_j(\mathbf{x})} \qquad \text{for } i = 1,2,$$

and

$$\mu_1(\mathbf{x}) = \mu_{11}(x_1)\mu_{21}(x_2), \quad \mu_2(\mathbf{x}) = \mu_{11}(x_1)\mu_{22}(x_2),$$

$$\mu_3(\mathbf{x}) = \mu_{12}(x_1)\mu_{21}(x_2), \quad \mu_4(\mathbf{x}) = \mu_{12}(x_1)\mu_{22}(x_2).$$

One sees from equation (12) that the fuzzy controller is actually a nonlinear PD controller with variable proportional-gain $b_1(\mathbf{x})$ and derivative-gain $b_2(\mathbf{x})$. The membership functions relevant for the stability determination are just four segments of straight lines covering the equilibrium point, although $\mu_{11}(x_1)$ and $\mu_{22}(x_2)$ are in triangular shape and $\mu_{12}(x_1)$ and $\mu_{21}(x_2)$ are in trapezoidal shape. As a result, we know, without any computation, that $\mu_i(\mathbf{x})$, $i = 1,2,3,4$, is a second-order polynomial in terms of $x_1(n)x_2(n)$; and consequently the numerator and denominator of $b_1(\mathbf{x})$ and $b_2(\mathbf{x})$ are second-order polynomials in terms of $x_1(n)x_2(n)$, too. Therefore, the result of first-order derivative of $u(n)$ with respect to $x_1(n)$ or $x_2(n)$ at the equilibrium point is unique. As we pointed out earlier, one can conclude that, in many cases, a fuzzy controller is linearizable about the equilibrium point even without seeing the actual mathematical formulation of the fuzzy controller. This example shows this point well and also demonstrates the practicality of our results.

We then compute:

$$\mu_1(0) = \mu_{11}(0)\cdot\mu_{21}(0) = 0.2182, \quad \mu_2(0) = \mu_{11}(0)\cdot\mu_{22}(0) = 0.4321,$$

$$\mu_3(0) = \mu_{12}(0)\cdot\mu_{21}(0) = 0.1429, \quad \mu_4(0) = \mu_{12}(0)\cdot\mu_{22}(0) = 0.2828,$$

and hence

$$b_1(0) = 18.0405 \quad \text{and} \quad b_2(0) = -5.4456.$$

The transfer function of the closed-loop fuzzy control system around the equilibrium point is found as follows:

$$\frac{\theta(z)}{\mathrm{SP}(z)} = \frac{-0.0549z + 0.0567}{z^2 - 2.0412z + 1.0533},$$

which has two poles $1.0206 \pm 0.1081i$. Both the poles are outside the unit circle and hence the fuzzy control system designed is asymptotically unstable at the equilibrium point. Here, SP(z) is the z-transform of SP(n).

The second example below relates to the first example.

Example 2. If the controller designer wants to keep all the components of the fuzzy controller in Example 1 unchanged except the values of a_{ij}'s, what should a_{ij}'s be in order to achieve asymptotic stability for the fuzzy control system around the equilibrium point?

Solution. We first design a linear PD controller with proportional-gain 5.0 and derivative-gain 2.0, which we know can control the pendulum with asymptotic stability at the equilibrium point. This means that $d_1 = 5$ and $d_2 = 2$ in (8) and we get two independent linear equations:

$$\mu_1(0)\,a_{11} + \mu_2(0)\,a_{12} + \mu_3(0)\,a_{13} + \mu_4(0)\,a_{14} = d_1$$

$$\mu_1(0)\,a_{21} + \mu_2(0)\,a_{22} + \mu_3(0)\,a_{23} + \mu_4(0)\,a_{24} = d_2$$

We let $a_{11} = a_{12} = a_{13} = a_{21} = a_{22} = a_{23} = 1$ and compute $a_{14} = 16.2207$ and $a_{24} = 4.8052$ using (9). The fuzzy control system should be locally stable with these new values of a_{ij}'s.

Indeed, the transfer function of the closed-loop fuzzy control system around the equilibrium point now becomes

$$\frac{\theta(z)}{\mathrm{SP}(z)} = \frac{0.02z - 0.0195}{z^2 - 1.9674z + 0.9777},$$

which has two poles $0.9837 \pm 0.1005i$ that are within the unit circle.

The last example below relates to the first two examples.

Example 3. Suppose that the mathematical model of the pendulum is unavailable but we know it is linearizable around the equilibrium point. Also, suppose that we know, after experimenting, that a linear PD controller with proportional-gain 5.0 and derivative-gain 2.0 can control the pendulum with asymptotic stability at the equilibrium point. How should the values of a_{ij}'s be chosen so that the fuzzy control system is locally stable?

Solution. The parameter computation is similar to that in Example 2. As one of the many possible results, the parameter values calculated in Example 2 can be used to achieve the desired local stability for the present example.

Example 3 shows that one only needs to know minimal information about the fuzzy controller and the plant model in order to design a locally stable fuzzy control system of the general class. The example clearly illustrates the practicality of our new design technique.

5. *Conclusion*

We have proved analytically that the general TS fuzzy controllers are nonlinear controllers with variable gains changing with input variables. Using Lyapunov's linearization method, we have established a necessary and sufficient condition for analytically determining asymptotic stability of the general TS control systems at the equilibrium point. The condition is simple and practical in that it can be used not only when the plant model is explicitly available but also when the model is unavailable. We have utilized the stability condition to design general TS fuzzy control systems, with or without plant models, that are stable at least around the equilibrium point. Three numerical examples are provided to illustrate in detail how to use the stability condition for the stability determination as well as for the design of stable TS fuzzy control systems.

Our results offer four important practical advantages: (1) our stability condition, being a necessary and sufficient one, is the tightest possible stability condition, (2) the condition is simple and easy to use partially because it only needs the fuzzy controller structure around the equilibrium point, (3) the condition can be used for determining system local stability and designing fuzzy control systems that are stable at least around the equilibrium point even when the explicit plant model is unavailable, and (4) the condition covers a very broad range of nonlinear TS fuzzy control systems, for which a meaningful global stability condition seems impossible to establish.

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