

General Analytical Structure of Typical Fuzzy Controllers and Their Limiting Structure Theorems*

HAO YING†

Key Words—Control system analysis; control theory; fuzzy control; fuzzy systems; nonlinear control systems; PID control; relay control.

Abstract—The fuzzy controllers studied in this paper have r input variables (error, rate change of error of process output, etc.). N members of input fuzzy sets are employed to fuzzify the inputs. N' nonlinear control rules represented by an arbitrary function f are used, which are evaluated by any type of fuzzy logic in conjunction with any inference method. The center of gravity algorithm is utilized for defuzzification. The author proves that the general analytical structure of the fuzzy controllers is the sum of a global f -dependent nonlinear controller and a local nonlinear controller which locally adjusts control action of the global controller. As $N \rightarrow +\infty$, the global nonlinear controller approaches a global f -dependent nonlinear controller while the local nonlinear controller disappears. A necessary and sufficient condition is obtained for judging convergence of limiting structure of the fuzzy controllers. If linear control rules are used, the global controller is a global r -dimensional multilevel relay which approaches a global linear controller as $N \rightarrow +\infty$. Two illustrative examples, which deal with linear, product, maximum and minimum control rules, are given.

1. Introduction

FUZZY CONTROLLERS are rule-based, which implement human operators' linguistic control strategies to control processes without their explicit mathematical models. Once a fuzzy controller is constructed, its analytical structure is uniquely determined. Such an analytical structure, however, is nonlinear in general and is hard to obtain. Nevertheless, it is essential to attain the analytical structure of fuzzy controllers before people can precisely understand why and how fuzzy controllers work. The analytical structures of a number of fuzzy controllers have been revealed in relation to nonfuzzy control theory. A fuzzy controller with two inputs, triangular-shaped input fuzzy sets and linear control rules was proven to be the sum of a global two-dimensional multilevel relay and a local nonlinear PI controller (Ying, 1993a) and a fuzzy controller with two inputs, trapezoidal-shaped input fuzzy sets and nonlinear control rules was proven to be the sum of a global control-rules-dependent nonlinear controller and a local nonlinear PI-like controller (Ying, 1992, 1993b). The objectives of this research are (1) to generalize such conclusions to much more general fuzzy

controllers; and (2) to explore limiting structures of these general fuzzy controllers as the number of members of input fuzzy sets approaches ∞ .

We previously showed that the fuzzy controllers with linear control rules approached a linear controller as the number of members of input fuzzy sets approached ∞ (Buckley and Ying, 1989). The results were extended to some more generally fuzzy controllers with linear control rules (Buckley, 1990) and to the fuzzy controllers with nonlinear control rules (Ying, 1993b). Recently, limiting structures of the fuzzy controllers using linear, product, maximum and minimum control rules were disclosed for some fuzzy controllers (Bouslama and Ichikawa, 1992).

In the next section, configuration of typical fuzzy controllers is defined. In Section 3, theoretical results on analytical and limiting structures of the fuzzy controllers are obtained to achieve the above-mentioned objectives. In Section 4, two illustrative examples are given.

2. Configuration of typical fuzzy controllers

The fuzzy controllers in this paper have r input variables. Each input variable, denoted as $e^{(i)}(k)$, is i th order difference of error of process output, $e(k)$ (k represents the sampling time). Values of the input variables are first scaled and the scaled inputs are then fed into the fuzzy controllers. The scaled inputs are represented as

$$x_i(k) = a_i e^{(i)}(k), \quad (2.1)$$

where

$$0 \leq i \leq r-1, \quad (2.2)$$

$$e(k) = y(k) - \text{setpoint}, \quad (2.3)$$

$$e^{(0)}(k) = e(k). \quad (2.4)$$

The $y(k)$ is the process output, setpoint is the desired process output and a_i s are input scalars for $e^{(i)}(k)$ s. Subscripts in the paper are always integers. Without losing generality, assume

$$-L \leq x_i(k) \leq L. \quad (2.5)$$

For input variable x_i , there exists an input fuzzy set \mathbf{X}_i , which has

$$N = 2J + 1 \quad (2.6)$$

members denoted as A_j ($-J \leq j \leq J$). Among N members, J members are for positive x_i , J members are for negative x_i , and one member is for near zero x_i . Membership functions of A_j s can literally be any shape but for the sake of easier mathematical manipulation in the proof of the main theoretical results later, some light restrictions are adopted. Central values of A_j s are required to be equally spaced with space

$$S = \frac{L}{J}. \quad (2.7)$$

* Received 25 July 1992; revised 6 November 1992; received in final form 25 November 1992. The original version of this paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Editor A. P. Sage. Corresponding author Ying's Fax # 409-772-6424; e-mail hying@beach.utmb.edu.

† Department of Physiology and Biophysics; Biomedical Engineering Center; and Office of Academic Computing, University of Texas Medical Branch, Galveston, TX 77555, U.S.A.

So, the central value of A_j is $j \cdot S$. Membership of A_j is zero at $(j - 1)S + \delta_j$ ($0 \leq \delta_j < S$) and increases to one at $j \cdot S - \epsilon_j$ ($0 \leq \epsilon_j < S - \delta_j$). The membership maintains a constant value of one between $j \cdot S - \epsilon_j$ and $j \cdot S + \lambda_j$ ($0 \leq \lambda_j < S$) and decreases from one at $j \cdot S + \lambda_j$ to zero at $(j + 1)S - \eta_j$ ($0 \leq \eta_j < S - \lambda_j$). The membership is zero elsewhere.

N^r control rules are necessary to cover all possible combinations of A_j s of X_j s. The control rules in this paper comply with the following law:

$$\text{IF } x_0 \text{ is } A_{p_0} \text{ AND } x_1 \text{ is } A_{p_1} \text{ AND } \dots \text{ AND } x_{r-1} \text{ is } A_{p_{r-1}} \\ \text{THEN } \Delta u \text{ is } \Delta U_m, \quad (2.8)$$

where ΔU_m is a member of output fuzzy set, designated as Y , and m is determined by an arbitrary nonlinear function f with integer output:

$$m = f(p_0, p_1, \dots, p_{r-1}). \quad (2.9)$$

Note m is constrained by

$$-M \leq m \leq M, \quad (2.10)$$

where

$$M = \text{Max} \{ |f(p_0, p_1, \dots, p_{r-1})|, -J \leq p_0, p_1, \dots, p_{r-1} \leq J \}. \quad (2.11)$$

In other words, there are $2M + 1$ ΔU_m s in Y (M ΔU_m s for positive Δu , M ΔU_m s for negative Δu and one ΔU_m for near zero Δu). Δu is crisp incremental output and is presumably subject to

$$-H \leq \Delta u \leq H. \quad (2.12)$$

Central values of ΔU_m s are equally spaced with space

$$V = \frac{H}{M}. \quad (2.13)$$

Thus, the central value of ΔU_m is $m \cdot V$. The shapes of the membership functions of ΔU_m s are required to be identical. Membership of ΔU_m is zero at $(m - 1)V + \sigma$ ($0 \leq \sigma < V$) and increases to one at $m \cdot V - \theta$ ($0 \leq \theta < V - \sigma$). The membership maintains constant value of one between $m \cdot V - \theta$ and $m \cdot V + \theta$ and decreases from one at $m \cdot V + \theta$ to zero at $(m + 1)V - \sigma$. The membership is zero elsewhere. It should be noted that membership of ΔU_m is symmetrical about its central value, $m \cdot V$.

Fuzzy logic used to evaluate ANDs in the control rules (2.8) can be any type, usually a T -norm such as Zadeh AND (Min) logic. If more than one control rule generates memberships for the same ΔU_m s, a fuzzy logic, often a T -co-norm like Zadeh OR (Max) logic, may be used to calculate the combined memberships (see Gupta and Qi, 1991 for a summary of commonly-used T -norms and T -co-norms). To reason ΔU_m s from $A_{p_0}, A_{p_1}, \dots, A_{p_{r-1}}$ in the control rules, an inference method is needed. Popular inference methods include Mamdani's minimum inference method, Larsen's product inference method and drastic product inference method (Mizumoto, 1988). In this paper, however, inference methods used are not limited to these three types. Any inference method can be used provided that it sensibly infers ΔU_m s from $A_{p_0}, A_{p_1}, \dots, A_{p_{r-1}}$.

The center of gravity algorithm is utilized for defuzzification. Since the shapes of the membership functions of ΔU_m s are identical and each membership function is symmetrical about its central value, the global centroid can be calculated from the local centroids which are the central values of ΔU_m s involved. Consequently, the crisp incremental output, scaled by an output scalar K , can be described as:

$$K \cdot \Delta u(k) = K \frac{\sum_{\mu_m(\Delta u) \neq 0} w(\mu_m(\Delta u)) \cdot m \cdot V}{\sum_{\mu_m(\Delta u) \neq 0} w(\mu_m(\Delta u))}, \quad (2.14)$$

where $w(\mu_m(\Delta u))$ s are the inference results, yielded by an inference method, on ΔU_m s. For Mamdani's minimum inference method or Larsen's product inference method, $w(\mu_m(\Delta u))$ s equal to the areas of ΔU_m s corresponding to the

combined memberships $\mu_m(\Delta u)$ s. For the drastic product inference method, $w(\mu_m(\Delta u))$ s equal to the combined memberships $\mu_m(\Delta u)$ s at the central values of ΔU_m s, $m \cdot V$.

The above-defined fuzzy controllers cover typical fuzzy controllers reported in fuzzy control literature. Therefore, the fuzzy controllers in this paper are typical ones in terms of fuzzification, fuzzy control rules, fuzzy logic, fuzzy inference method and defuzzification. The fuzzy controllers referred to thereafter are these typical fuzzy controllers.

3. Theoretical results on analytical structure of fuzzy controllers

Theorem 1 (general analytical structure theorem). The general analytical structure of typical fuzzy controllers is the sum of a global f -dependent nonlinear controller (denoted as $\Delta u_G(k)$) and a local nonlinear controller (denoted as $\Delta u_L(k)$) which locally adjusts control action of the global controller. More specifically,

$$K \cdot \Delta u(k) = \Delta u_G(k) + \Delta u_L(k), \quad (3.1)$$

where

$$\Delta u_G(k) = n_c \frac{KH}{M}, \quad (3.2)$$

$$\Delta u_L(k) = \frac{KH}{M} \frac{\sum_{h=1}^q w(\mu_{c_h}(\Delta u))(n_h - n_c)}{\sum_{h=1}^q w(\mu_{c_h}(\Delta u))}, \quad (3.3)$$

$$-M \leq n_h, n_c \leq M \text{ and } 1 \leq q \leq 2M + 1. \quad (3.4)$$

The $\mu_{c_h}(\Delta u)$ ($h = 1, 2, \dots, q$) represents the combined membership for ΔU_{n_h} .

Proof. Without losing generality, assume

$$p_i \cdot S \leq x_i(k) \leq (p_i + 1)S, \quad (3.5)$$

where

$$-J \leq p_i \leq J - 1. \quad (3.6)$$

After defuzzification, two memberships (one or both of which may be zero, depending on the definitions of the membership functions of A_j s and the values of $x_i(k)$ s)

$$\mu_{p_i}(x_i) \text{ and } \mu_{p_i+1}(x_i), \quad (3.7)$$

are attained for each of $x_i(k)$ s. Denote

$$\Delta x_i = x_i - (p_i + 0.5)S, \quad (3.8)$$

where $(p_i + 0.5)S$ is the center of the interval $[p_i \cdot S, (p_i + 1)S]$, it is obvious that

$$\mu_{p_i}(x_i) = \mu_{p_i}(\Delta x_i + (p_i + 0.5)S), \\ \mu_{p_i+1}(x_i) = \mu_{p_i+1}(\Delta x_i + (p_i + 0.5)S), \quad (3.9)$$

which means the memberships are functions of Δx_i with respect to $(p_i + 0.5)S$. For r inputs, at most $2r$ nonzero memberships are obtained, generating maximum 2^r different combinations of the memberships of $A_{p_0}, A_{p_0+1}, A_{p_1}, A_{p_1+1}, \dots, A_{p_{r-1}}$ and $A_{p_{r-1}+1}$. Consequently, up to 2^r control rules are activated, which produce at most 2^r nonzero memberships for some or all of ΔU_m s as the results of operation of a fuzzy logic (designated as FL) on ANDs in the control rules. That is

$$\mu_1(\Delta u) = \text{FL}(\mu_{p_0}(\Delta x_0), \mu_{p_1}(\Delta x_1), \dots, \mu_{p_{r-1}}(\Delta x_{r-1})) \\ \text{for } \Delta U_{m_1}, \quad (3.10)$$

$$\mu_2(\Delta u) = \text{FL}(\mu_{p_0+1}(\Delta x_0), \mu_{p_1}(\Delta x_1), \dots, \mu_{p_{r-1}}(\Delta x_{r-1})) \\ \text{for } \Delta U_{m_2}, \quad (3.11)$$

⋮

$$\mu_{2^r}(\Delta u) = \text{FL}(\mu_{p_0+1}(\Delta x_0), \mu_{p_1+1}(\Delta x_1), \dots, \mu_{p_{r-1}+1}(\Delta x_{r-1})) \\ \text{for } \Delta U_{m_{2^r}}. \quad (3.12)$$

Since some of the memberships are possibly for the same

ΔU_m s, a fuzzy logic may be needed to produce the combined memberships for the same ΔU_m s. After use of the fuzzy logic, supposedly there exist q ($1 \leq q \leq 2M + 1$) distinctively different ΔU_m s with the following nonzero combined memberships:

$$\mu_{c_1}(\Delta u) \text{ for } \Delta U_{n_1}, \quad (3.13)$$

$$\mu_{c_2}(\Delta u) \text{ for } \Delta U_{n_2}, \quad (3.14)$$

$$\vdots$$

$$\mu_{c_q}(\Delta u) \text{ for } \Delta U_{n_q}, \quad (3.15)$$

where

$$-M \leq n_1, n_2, \dots, n_q \leq M. \quad (3.16)$$

Substituting the combined memberships into the defuzzification algorithm (2.14) yields

$$\begin{aligned} K \cdot \Delta u(k) &= K \frac{\sum_{h=1}^q w(\mu_{c_h}(\Delta u)) \cdot n_h \cdot V}{\sum_{h=1}^q w(\mu_{c_h}(\Delta u))} \\ &= n_\zeta KV + K \frac{\sum_{h=1}^q w(\mu_{c_h}(\Delta u))(n_h - n_\zeta)V}{\sum_{h=1}^q w(\mu_{c_h}(\Delta u))} \\ &= n_\zeta \frac{KH}{M} + \frac{KH}{M} \frac{\sum_{h=1}^q w(\mu_{c_h}(\Delta u))(n_h - n_\zeta)}{\sum_{h=1}^q w(\mu_{c_h}(\Delta u))}, \quad (3.17) \end{aligned}$$

where ζ can be any integer but $1 \leq \zeta \leq q$.

The first part of (3.17) is named $\Delta u_G(k)$ in (3.2) while the second part is called $\Delta u_L(k)$ in (3.3). $\Delta u_G(k)$ is a global f -dependent nonlinear controller because its control action is determined by n_ζ which is a nonlinear function (f) of p_0, p_1, \dots, p_{r-1} with respect to the origin of the scaled input space. $\Delta u_L(k)$ is a local nonlinear controller since its control action is ultimately determined by Δx_s with respect to the center of the r -dimensional cube configured by the r intervals in (3.5), $((p_0 + 0.5)S, (p_1 + 0.5)S, \dots, (p_{r-1} + 0.5)S)$. Note that n_h and n_ζ remain the same constants if x_s do not jump from one cube to another cube from one sampling time to another sampling time. Hence, as long as x_s stay in the same cube, control action from the global controller remains unchanged. On the other hand, control action from the local controller changes as Δx_s change. Hence, it is clear that the role of the local nonlinear controller is to fine adjust locally control action of the global nonlinear controller. \square

It is apparent that Theorem 1 still holds if a fuzzy logic is not used to get the combined memberships for the same members of ΔU_m s.

Given a specific configuration of a fuzzy controller, finding an explicit analytical structure of $\Delta u_L(k)$ is often a difficult task. It is, however, easier to derive an explicit analytical structure of $\Delta u_G(k)$ once f is known. The following theorem uncovers the analytical structure of $\Delta u_G(k)$ when linear control rules are used.

Theorem 2 (analytical structure theorem for linear control rules). If linear control rules represented by a linear function

$$f(p_0, p_1, \dots, p_{r-1}) = \sum_{i=0}^{r-1} b_i p_i + b_r, \quad (3.18)$$

are used, $\Delta u_G(k)$ is a global r -dimensional multilevel relay with respect to the integer inputs, p_s :

$$\Delta u_G(k) = C_1 \left[\sum_{i=0}^{r-1} b_i p_i + b_r + C_2 \right], \quad (3.19)$$

where C_1 and C_2 are constants described in (3.23).

Proof.

$$\begin{aligned} M &= \text{Max} \left\{ \left[\sum_{i=0}^{r-1} b_i p_i + b_r \right], -J \leq p_i \leq J \right\} \\ &= J \sum_{i=0}^{r-1} |b_i| + |b_r|. \quad (3.20) \end{aligned}$$

The n_ζ in (3.2) can be expressed as

$$n_\zeta = \sum_{i=0}^{r-1} b_i(p_i + \psi_i) + b_r, \quad (3.21)$$

where ψ_i is a constant, either 0 or 1, for a given i . Thus

$$\begin{aligned} \Delta u_G(k) &= n_\zeta \frac{KH}{M} = \frac{KH}{J \sum_{i=0}^{r-1} |b_i| + |b_r|} \left[\sum_{i=0}^{r-1} b_i(p_i + \psi_i) + b_r \right] \\ &= C_1 \left[\sum_{i=0}^{r-1} b_i p_i + b_r + C_2 \right], \quad (3.22) \end{aligned}$$

where

$$C_1 = \frac{KH}{J \sum_{i=0}^{r-1} |b_i| + |b_r|} \quad \text{and} \quad C_2 = \sum_{i=0}^{r-1} b_i \psi_i. \quad (3.23)$$

Note C_1 and C_2 are constants and the value of the linear function (3.18) at p_s is an integer. Therefore, $\Delta u_G(k)$ is a global r -dimensional multilevel relay with respect to its integer input, p_s . \square

It is of theoretical and practical interest to predict what will happen to the structures of $\Delta u_G(k)$ and $\Delta u_L(k)$ as N (therefore, J) grows without bound. The following theorem deals with this issue.

Theorem 3 (general limiting structure theorem).

$$(1) \quad \lim_{N \rightarrow +\infty} \Delta u_L(k) = 0, \quad (3.24)$$

$$\lim_{N \rightarrow +\infty} K \cdot \Delta u(k) = \lim_{N \rightarrow +\infty} \Delta u_G(k) = KH \cdot \lim_{N \rightarrow +\infty} \frac{n_\zeta}{M} \neq 0, \quad (3.25)$$

if and only if

$$\lim_{N \rightarrow +\infty} \frac{n_h - n_\zeta}{M} = 0, \quad \text{for } h = 1, 2, \dots, q. \quad (3.26)$$

(2) The convergence rate of $\Delta u_L(k) \rightarrow 0$ is inversely proportional to M .

Proof. (1) Note $w(\mu_{c_h}(\Delta u))$ remains nonzero when $N \rightarrow +\infty$. If, and only if, f satisfies the condition (3.26), then

$$\begin{aligned} \lim_{N \rightarrow +\infty} \Delta u_L(k) &= \lim_{N \rightarrow +\infty} \frac{KH \sum_{h=1}^q w(\mu_{c_h}(\Delta u))(n_h - n_\zeta)}{\sum_{h=1}^q w(\mu_{c_h}(\Delta u))} \\ &= KH \cdot \lim_{N \rightarrow +\infty} \frac{\sum_{h=1}^q w(\mu_{c_h}(\Delta u)) \frac{n_h - n_\zeta}{M}}{\sum_{h=1}^q w(\mu_{c_h}(\Delta u))} = 0. \quad (3.27) \end{aligned}$$

As a consequence,

$$\lim_{N \rightarrow +\infty} K \cdot \Delta u(k) = \lim_{N \rightarrow +\infty} \Delta u_G(k) = KH \cdot \lim_{N \rightarrow +\infty} \frac{n_\zeta}{M} \neq 0. \quad (3.28)$$

The expression (3.28) is not equal to zero because, as $N \rightarrow +\infty$, the output of fuzzy controllers will not be constantly zero with respect to the input $x_i(k)$ s.

If, on the other hand, there exists at least one h ($1 \leq h \leq q$) which makes

$$\lim_{N \rightarrow +\infty} \frac{n_h - n_\zeta}{M} \neq 0, \quad (3.29)$$

then

$$\lim_{N \rightarrow +\infty} \Delta u_L(k) \neq 0. \quad (3.30)$$

(2) According to (3.27), the convergence rate of $\Delta u_L(k) \rightarrow 0$ is obviously inversely proportional to M . \square

The following theorem reveals the limiting structure of the r -dimensional multilevel relay (3.19) representing $\Delta u_G(k)$ of typical fuzzy controllers with linear control rules.

Theorem 4 (limiting structure theorem for linear control rules). The limiting structure of typical fuzzy controllers with linear control rules represented in (3.18) is a linear controller:

$$\lim_{N \rightarrow +\infty} K \cdot \Delta u(k) = \frac{KH}{L \sum_{i=0}^{r-1} |b_i|} \sum_{i=0}^{r-1} a_i b_i e^{(i)}(k). \quad (3.31)$$

Proof. To determine convergence of limiting structure of the fuzzy controllers using the linear control rules, the necessary and sufficient condition (3.26) needs to be checked first. The n_h in (3.26) can be represented by

$$n_h = \sum_{i=0}^{r-1} b_i(p_i + \varphi_i) + b_r, \quad (3.32)$$

where φ_i is a constant, either 0 or 1, for a given i . The n_z and M in (3.26) were expressed in (3.21) and (3.20), respectively. Hence

$$\lim_{N \rightarrow +\infty} \frac{n_h - n_z}{M} = \lim_{N \rightarrow +\infty} \frac{\sum_{i=0}^{r-1} b_i(\varphi_i - \psi_i)}{J \sum_{i=0}^{r-1} |b_i| + |b_r|} = 0, \quad (3.33)$$

which means the condition (3.26) is satisfied. As a result,

$$\begin{aligned} \lim_{N \rightarrow +\infty} K \cdot \Delta u(k) &= KH \cdot \lim_{N \rightarrow +\infty} \frac{n_z}{M} \\ &= KH \cdot \lim_{N \rightarrow +\infty} \frac{\sum_{i=0}^{r-1} b_i(p_i + \psi_i) + b_r}{J \sum_{i=0}^{r-1} |b_i| + |b_r|} \\ &= \frac{KH}{\sum_{i=0}^{r-1} |b_i|} \lim_{N \rightarrow +\infty} \sum_{i=0}^{r-1} b_i \frac{p_i + \psi_i}{J}. \end{aligned} \quad (3.34)$$

Because of (3.5), we have

$$\frac{p_i}{J} \leq \frac{a_i e^{(i)}(k)}{L} \leq \frac{p_i + 1}{J}, \quad (3.35)$$

which means

$$\lim_{N \rightarrow +\infty} \frac{p_i}{J} = \lim_{N \rightarrow +\infty} \frac{p_i + 1}{J} = \frac{a_i e^{(i)}(k)}{L}. \quad (3.36)$$

Thus, (3.34) becomes

$$\lim_{N \rightarrow +\infty} K \cdot \Delta u(k) = \frac{KH}{L \sum_{i=0}^{r-1} |b_i|} \sum_{i=0}^{r-1} a_i b_i e^{(i)}(k). \quad (3.37)$$

The proof is completed.

Recall that Theorem 3 reveals that if the necessary and sufficient condition (3.26) is satisfied then (1) the local nonlinear controller, $\Delta u_L(k)$, disappears as $N \rightarrow +\infty$, and (2) the limiting structure of typical fuzzy controllers is a unique global nonlinear controller fundamentally determined by the control rules (f). An important question arises: 'Can the global f -dependent nonlinear controller, $\Delta u_G(k)$, approach any given function as $N \rightarrow +\infty$?' The answer is no, as the following theorem proves.

Theorem 5 (limit approach theorem for limiting

structure). The limiting structure of $\Delta u_G(k)$ cannot always approach a given function, even when the necessary and sufficient condition (3.26) is met.

Proof. Take a simple example for the purpose of proof. Suppose we want the limiting structure of $\Delta u_G(k)$ to be a linear function

$$g(x_0, x_1, \dots, x_{r-1}) = \sum_{i=0}^{r-1} d_i x_i + d_r, \quad d_r \neq 0. \quad (3.38)$$

As shown in Theorem 4, fuzzy controllers using linear control rules represented in (3.18) can only approach the linear controller

$$\sum_{i=0}^{r-1} d_i x_i, \quad (3.39)$$

where

$$d_i = \frac{b_i KH}{L \sum_{i=0}^{r-1} |b_i|}. \quad (3.40)$$

Obviously, the d_r term in (3.38) cannot be approached. \square

In the next section, explicit $\Delta u_G(k)$, $\Delta u_L(k)$ and limiting structure of some example fuzzy controllers will be shown.

4. Illustrative examples

Example 1. We first investigate a fuzzy controller using linear control rules. Assume the fuzzy controller employs input fuzzy sets, any symmetrical-shaped output fuzzy sets, Zadeh fuzzy logic AND, Lukasiewicz fuzzy logic OR, Larsen's product inference method (or drastic product inference method) and a center of gravity defuzzification algorithm. According to Theorem 2, the global nonlinear controller, $\Delta u_G(k)$, should be a global two-dimensional multilevel relay. The local nonlinear controller, $\Delta u_L(k)$, is a nonlinear PI controller with variable proportional-gain and integral-gain (Ying, 1993a). Based on Theorem 4, the limiting structure of this fuzzy controller should obviously be a global linear PI controller.

It should be pointed out that the local controller becomes a nonlinear PI-like controller with variable gains if the fuzzy controller uses trapezoidal-shaped input fuzzy sets instead of the triangular-shaped ones (Ying, 1992).

Example 2. We study the limiting structure of the fuzzy controller in the above example when the following nonlinear control rules are employed:

(1) product control rules represented by the function

$$f(p_0, p_1, \dots, p_{r-1}) = \prod_{i=0}^{r-1} p_i, \quad (4.1)$$

(2) maximum or minimum control rules represented by the functions

$$f(p_0, p_1, \dots, p_{r-1}) = \text{Max}\{p_i, 0 < i < r-1\}. \quad (4.2)$$

$$f(p_0, p_1, \dots, p_{r-1}) = \text{Min}\{p_i, 0 \leq i \leq r-1\}. \quad (4.3)$$

When the product control rules are used,

$$M = J^r, \quad (4.4)$$

$$\begin{aligned} \lim_{N \rightarrow +\infty} \frac{n_h - n_z}{M} &= \lim_{N \rightarrow +\infty} \frac{\prod_{i=0}^{r-1} (p_i + \varphi_i) - \prod_{i=0}^{r-1} (p_i + \psi_i)}{J^r} \\ &= \frac{1}{L^r} \prod_{i=0}^{r-1} a_i e^{(i)}(k) - \frac{1}{L^r} \prod_{i=0}^{r-1} a_i e^{(i)}(k) = 0. \end{aligned} \quad (4.5)$$

That is, the condition (3.26) is met. Hence,

$$\lim_{N \rightarrow +\infty} K \cdot \Delta u(k) = KH \cdot \lim_{N \rightarrow +\infty} \prod_{i=0}^{r-1} \frac{p_i + \psi_i}{J^r} = \frac{KH}{L^r} \prod_{i=0}^{r-1} a_i e^{(i)}(k). \quad (4.6)$$

When the maximum or minimum control rules are used,

$$M = J, \quad (4.7)$$

$$\lim_{N \rightarrow +\infty} \frac{n_h - n_r}{M} = \lim_{N \rightarrow +\infty} \frac{\text{Max}\{p_i + \varphi_i\} - \text{Max}\{p_i + \psi_i\}}{J} = 0, \quad (4.8)$$

$$\lim_{N \rightarrow +\infty} \frac{n_h - n_r}{M} = \lim_{N \rightarrow +\infty} \frac{\text{Min}\{p_i + \varphi_i\} - \text{Min}\{p_i + \psi_i\}}{J} = 0, \quad (4.9)$$

which mean the condition (3.26) is satisfied. So,

$$\begin{aligned} \lim_{N \rightarrow +\infty} K \cdot \Delta u(k) &= KH \cdot \lim_{N \rightarrow +\infty} \frac{\text{Max}\{p_i + \psi_i\}}{J} \\ &= KH \cdot \text{Max} \left\{ \lim_{N \rightarrow +\infty} \frac{p_i + \psi_i}{J} \right\} \\ &= \frac{KH}{L} \text{Max} \{a_i e^{(i)}(k), 0 \leq i \leq r-1\}, \quad (4.10) \end{aligned}$$

$$\begin{aligned} \lim_{N \rightarrow +\infty} K \cdot \Delta u(k) &= KH \cdot \lim_{N \rightarrow +\infty} \frac{\text{Min}\{p_i + \psi_i\}}{J} \\ &= KH \cdot \text{Min} \left\{ \lim_{N \rightarrow +\infty} \frac{p_i + \psi_i}{J} \right\} \\ &= \frac{KH}{L} \text{Min} \{a_i e^{(i)}(k), 0 \leq i \leq r-1\}. \quad (4.11) \end{aligned}$$

It should be noted that Bouslama and Ichikawa (1992) obtained similar limiting structure results for some fuzzy controllers employing these control rules. The results presented in this example, however, are much more general because they cover virtually any fuzzy controller using the control rules.

5. Conclusions

It has been proven in this paper that the general analytical structure of typical fuzzy controllers, which use any type of fuzzification for inputs, any type of control rules, any fuzzy logic, any inference method and the center of gravity defuzzification algorithm, is the sum of a global control-rules-dependent nonlinear controller and a local nonlinear controller locally tuning control action of the global controller. The structure of the global nonlinear controller is fundamentally determined by control rules alone while the structure of the local nonlinear controller is determined by

all the components of fuzzy controllers. The global nonlinear controller plays a global role in fuzzy control and it dominates control action of the fuzzy controllers, especially when N is large. The local nonlinear controller plays a local but important role in fuzzy control. The larger the N is, the less the role of the local controller. In an extreme, such a role disappears as $N \rightarrow +\infty$. It is mainly such peculiar structural configuration of the fuzzy controllers as well as characteristics of the local nonlinear controller that differentiates, in terms of analytical structure, fuzzy controllers from other conventional nonlinear controllers. The general analytical structure of the fuzzy controllers helps explain why and how the fuzzy controllers work. Further investigation on characteristics of the global and local nonlinear controllers is necessary in order to advance fuzzy control technology.

Acknowledgement The author would like to thank Prof. Louis C. Sheppard for his encouragement and support in the research.

References

- Buckley, J. J. (1990). Further limit theorems for linear control rules. *Fuzzy Sets and Systems*, **36**, 225-233.
- Buckley, J. J. and H. Ying (1989). Fuzzy controller theory: limit theorems for linear fuzzy control rules. *Automatica*, **25**, 469-472.
- Bouslama, F. and A. Ichikawa (1992). Fuzzy control rules and their natural control laws. *Fuzzy Sets and Systems*, **48**, 65-86.
- Gupta, M. M. and J. Qi (1991). Theory of T -norms and fuzzy inference methods. *Fuzzy Sets and Systems*, **40**, 431-450.
- Mizumoto, M. (1988). Fuzzy controls under various fuzzy reasoning methods. *Information Sciences*, **45**, 129-151.
- Ying, H. (1992). A fuzzy controller with nonlinear control rules is the sum of a global nonlinear controller and a local nonlinear PI-like controller. *Proc. of 1992 NASA Int. Joint Technology Workshop on Fuzzy Logic and Neural Network*, Houston, TX, U.S.A., 1-3 June, 1992.
- Ying, H. (1993a). A nonlinear fuzzy controller with linear control rules is the sum of a global two-dimensional multilevel relay and a local nonlinear proportional-integral controller. *Automatica*, **29**, 499-505.
- Ying, H. (1993b). Analytical analysis of structure fuzzy controllers. *IEEE Trans. Aut. Control* (submitted).