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Brief Paper

# Fuzzy Controller Theory: Limit Theorems for Linear Fuzzy Control Rules*

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**Key Words**—Fuzzy control.

**Abstract**—For a general fuzzy controller employing linear fuzzy control rules it is shown that as the number of rules grow the defuzzified output becomes a linear function of the input. In particular, for a sufficiently large number of rules, the defuzzified output is approximately the same as the PI (PID) controller.

## 1. *Introduction*

IN THIS SECTION we first review the basic components of a fuzzy controller and set up the notation needed to establish our main results in Section 2. Section 3 contains an example illustrating linear fuzzy control rules. The last section contains a brief summary and a discussion of the implications of our results.

Consider a process with one output $y(t)$, one input $u(t)$, and set point $s$ which is the desired process output. From our observations on $y(t)$ the controller computes (approximates) $y^{(i)}(t)$ which is the $i$th derivative of error which is equal to $y(t) - s$, for $0 \leq i \leq n$. The 0th derivative of error is just equal to error. We choose constants $c_i$, $0 \leq i \leq n$, so that $-1 \leq c_i y^{(i)}(t) \leq 1$ for $t \geq 0$. Let $r^{(i)}(t) = c_i y^{(i)}(t)$, $0 \leq i \leq n$. We will assume the input to the fuzzy controller is the $r^{(i)}(t)$, $0 \leq i \leq n$, the values of which are always in the interval $[-1,1]$.

The defuzzified output from the fuzzy controller will be denoted by $\delta(t)$ the values of which will also be in $[-1, 1]$ for $t \geq 0$. Let $\Delta$ be the sampling period and $k\Delta$ be the time at which we observe the process and compute the $r^{(i)}(t)$ and have the fuzzy controller produce $\delta(t)$. Then input into the controlled system will be

$$u(t) = u(t - \Delta) + (\delta(t))(\Delta) \quad \text{at} \quad t = \Delta, 2\Delta, \ldots.$$

The fuzzy controller consists of three main parts: (1) the fuzzification process, or the fuzzy numbers used to define the linguistic variables in the fuzzy control rules; (2) the fuzzy control rules and the procedure employed to evaluate these rules; and (3) the defuzzification algorithm. We now discuss each of these components of the fuzzy controller in detail.

1.1. *Fuzzy numbers.* We will need $2N + 1$, $N \geq 1$, fuzzy numbers for each of the $n + 1$ inputs $r^{(i)}$ to the fuzzy controller. The fuzzy numbers for input $r^{(i)}$ will be denoted by $R_j^{(i)}$, for $1 \leq j \leq 2N + 1$. Let $v(j) = [j - (N + 1)]/N$ which will be the central value of $R_j^{(i)}$. The graph of the membership function $y = \mu(x \mid R_j^{(i)})$ of $R_j^{(i)}$, $2 \leq j \leq 2N$ is: (1) zero outside $(v(j - 1), v(j + 1))$; (2) one at $v(j)$; (3) a positive continuous non-decreasing function of $x$ on $(v(j - 1), v(j)]$; and (4) a positive continuous non-increasing function of $x$ on $[v(j), v(j + 1))$. The graph of $y = \mu(x \mid R_j^{(i)})$ may be a triangle, a trapezoid, or a non-linear "bell" shaped curve over $(v(j - 1), v(j + 1))$ for $2 \leq j \leq 2N$. The graph of $y = \mu(x \mid R_1^{(i)})$ is a positive continuous non-increasing function of $x$ on $[-1, v(2))$ from equal to one at $x = -1$ and zero for $x \geq v(2)$. The graph of $y = \mu(x \mid R_{2N+1}^{(i)})$ is a positive continuous non-decreasing function of $x$ on $(v(2N), 1]$ from zero for $x < v(2N)$ to equal to one at $x = 1$. All fuzzy numbers have their support in $[-1, 1]$. We have $N$ negative, $N$ positive, and one "zero" fuzzy number for each input.

Given a value of $r^{(i)}$ in $[-1, 1]$ there is a unique value of $j$ in $\{1, 2, \ldots, 2N\}$, say $j(i)$, so that $r^{(i)}$ is in the interval $[v(j(i)), v(j(i) + 1))$ if $1 \leq j(i) < 2N$, or $r^{(i)}$ is in $[1 - N^{-1}, 1]$ if $j(i) = 2N$. Let $\mu_j^{(i)} = \mu(r^{(i)} \mid R_j^{(i)})$. For each $i$ only two $\mu_j^{(i)}$ can possibly be nonzero and these two occur when $j = j(i)$ and $j = j(i) + 1$.

1.2. *Rules.* We will have fuzzy control rules $\mathscr{R}_k$ for $1 \leq k \leq K = (n + 1)(2N) + 1$. Let $\mathscr{S}^{(i)}$ stand for some statement about $r^{(i)}$, $0 \leq i \leq n$. For example, $\mathscr{S}^{(0)}$ could be "error" and $\mathscr{S}^{(1)}$ might represent "rate" where "rate" stands for the rate of change of error. Then the first and last fuzzy control rules are

$$\mathscr{R}_1\text{: If } [\mathscr{S}^{(0)} = R_1^{(0)}] \text{ AND} \cdots \text{AND } [\mathscr{S}^{(n)} = R_1^{(n)}], \quad \text{then} \quad O = O_K$$

and

$$\mathscr{R}_K\text{: If } [\mathscr{S}^{(0)} = R_{2N+1}^{(0)}] \text{ AND} \cdots \text{AND } [\mathscr{S}^{(n)} = R_{2N+1}^{(n)}], \quad \text{then} \quad O = O_1$$

where $O$ denotes output. Output is a discrete fuzzy set the elements of which $O_l$, $1 \leq l \leq K$, are fuzzy numbers. Traditionally, one uses linguistic variables like "negative-large", "positive-very small", etc. in place of the $R_j^{(i)}$ and the $O_l$ in the fuzzy control rules. However, we will employ the fuzzy numbers $R_j^{(i)}$ and $O_l$ themselves, which define these linguistic variables, in the fuzzy control rules.

The members $O_l$, $1 \leq l \leq K$, of $O$ are fuzzy numbers similar to those used for the inputs. Let

$$w(l) = \frac{l - [(n + 1)N + 1]}{(n + 1)N} \tag{1}$$

which will be the central value of $O_l$, $1 \leq l \leq K$. The graph of the membership function $y = \mu(x \mid O_l)$ of $O_l$, $2 \leq l \leq K - 1$, is: (1) zero outside $(w(l - 1), w(l + 1))$; (2) one at $w(l)$; (3) a continuous monotonically increasing function of $x$ on $[w(l - 1), w(l)]$ from zero at $x = w(l - 1)$ to one at $x = w(l)$; and (4) a continuous monotonically decreasing function of $x$ on $[w(l), w(l + 1)]$ from one at $x = w(l)$ to zero at $x = w(l + 1)$. The graph of $y = \mu(x \mid O_1)$ is a continuous monotonically decreasing function of $x$ on $[-1, w(2)]$ from one at $x = -1$ to zero at $x = w(2)$ and remains zero for $x > w(2)$. The graph of $y = \mu(x \mid O_K)$ is a continuous monotonically increasing function of $x$ on $[w(K - 1), 1]$ from zero at $x = w(K - 1)$ to one at $x = 1$ and $\mu(x \mid O_K) = 0$ for $x < w(K - 1)$. All the $O_l$ have their support in $[-1, 1]$ and there are $(n + 1)N$ negative, $(n + 1)N$ positive, and one "zero" fuzzy number in $O$.

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If $\mathscr{C}_j$, $1 \le j \le J$, denotes a clause like $[\mathscr{S}^{(i)} = R_j^{(i)}]$, then OR $\{\mathscr{C}_j \mid 1 \le j \le J\}$ stands for

$$\mathscr{C}_1 \text{ OR } \mathscr{C}_2 \text{ OR} \cdots \text{OR } \mathscr{C}_J. \tag{2}$$

We may now define all the other fuzzy control rules as

$$\mathscr{R}_k\text{: OR } \{[\mathscr{S}^{(0)} = R_{i_0}^{(0)}] \text{ AND} \cdots \text{AND } [\mathscr{S}^{(n)} = R_{i_n}^{(n)}] \mid i_0 + \cdots + i_n = n + k\}, \quad \text{then} \quad O = O_{K-k+1}$$

for $1 < k < K$. We say the rules numbered $k$, $1 < k < K$, are linear because the OR is taken over all subscripts with constant sum $n + k$, where $k$ is the rule number. Also notice that each fuzzy control rule has a distinct conclusion. There are $K$ fuzzy control rules and $O$ has $K$ distinct, and equally spaced, fuzzy numbers in $[-1, 1]$. All rules fire given the inputs $r^{(i)}$.

To evaluate a rule given the $r^{(i)}$ we must first evaluate all the clauses $[\mathscr{S}^{(i)} = R_j^{(i)}]$. The value of $[\mathscr{S}^{(i)} = R_j^{(i)}]$ is $\mu_j^{(i)} = \mu(r^{(i)} \mid R_j^{(i)})$. Let $T$ be any $t$-norm extended (by associativity) to $n + 1$ arguments and let $C$ be any co-$t$-norm extended (by associativity) to $n + 1$ arguments. We have a special case when $n = 0$ and then we set $T(\mu) = \mu$ and $C(\mu) = \mu$ for the $t$-norm and co-$t$-norm, respectively, of only one argument.

When $\mathscr{R}_k$ executes the value of its left-hand side is $\Delta_k$ which is given by

$$\Delta_1 = T(\mu_1^{(0)}, \ldots, \mu_1^{(n)}) \tag{3}$$

$$\Delta_K = T(\mu_{2N+1}^{(0)}, \ldots, \mu_{2N+1}^{(n)}) \tag{4}$$

and

$$\Delta_k = C\{T(\mu_{i_0}^{(0)}, \ldots, \mu_{i_n}^{(n)}) \mid i_0 + \cdots + i_n = n + k\} \tag{5}$$

for $1 < k < K$. We have employed the notation $C\{z_j \mid 1 \le j \le J\} = C(z_1, \ldots, z_J)$. Recall that at most two of the $\mu_j^{(i)}$, $1 \le j \le 2N + 1$, can be positive for each $i$. One may use different $t$-norms $T$ and different co-$t$-norms $C$ in equations (3)–(5). For example you could have $T$ = probabilistic AND in equations (3) and (4), $T = \min$ for equation (5), and $C$ = Lukasiewicz OR in equation (5) (Siler and Ying, 1989).

The numbers $\Delta_k$ produce the discrete fuzzy set $O$ = output the elements of which are $O_l$, $1 \le l \le K$. The membership value of $O_l$ is $\Delta_{K-l+1}$, $1 \le l \le K$.

1.3. *Defuzzify*. We will first consider two specific methods of defuzzifying $O$ into a real number $\delta \in [-1, 1]$ for input into the process. The first method employs the central value of each fuzzy number in $O$ and we define

$$\delta_0 = \frac{\sum_{l=1}^{K} \Delta_l w(K - l + 1)}{\sum_{l=1}^{K} \Delta_l}. \tag{6}$$

Let

$$P_l = \frac{\Delta_l}{\sum_{l=1}^{K} \Delta_l} \tag{7}$$

for $1 \le l \le K$. The $P_l$ are probabilities defined on the central values of the $O_l$ so $\delta_0$ is an expected output given by

$$\delta_0 = \sum_{l=1}^{K} P_l w(K - l + 1). \tag{8}$$

The second procedure uses a number $\alpha_l$ in $[w(l - 1), w(l + 1)]$ if $2 \le l \le K - 1$, or $\alpha_1$ in $[-1, w(2)]$, or $\alpha_K$ in $[w(K - 1), 1]$, so that $\mu(\alpha_l \mid O_l) = \Delta_l$ in place of the central values in equation (6). There are two possible values of $\alpha_l$, $1 \le l \le K - 1$, one to the left of the central value and one to the right of the central value, so that $\mu(\alpha_l \mid O_l) = \Delta_l$. It does not matter, for our results in the next section, which value for $\alpha_l$ is chosen in the following definition of $\delta_1$. The second defuzzifier is defined by

$$\delta_1 = \frac{\sum_{l=1}^{K} \alpha_l \Delta_l}{\sum_{l=1}^{K} \Delta_l} = \sum_{l=1}^{K} P_l \alpha_l \tag{9}$$

which is also an expected value calculation. However, now the $\alpha_l$ will change if the $\Delta_l$ are changed.

Our last defuzzifier $\delta_2$ is actually a general description of any "reasonable" defuzzifier that could be employed with the fuzzy controller described above. Given the $r^{(i)}$ in $[-1, 1]$, their values determine the $j(i)$ in $\{1, 2, \ldots, 2N\}$, and let

$$m = \sum_{j=0}^{n} j(i) - n.$$

The $\Delta_l$ can be positive only for $l = m, m + 1, \ldots, m + n + 1$ since, for each $i$, $\mu_j^{(i)}$ can be positive only for $j = j(i)$ and $j = j(i) + 1$. That is, we know $\Delta_l = 0$ for $1 \le l < m$ and $m + n + 1 < l \le K$. Which $\Delta_{m+j}$, $0 \le j \le n + 1$, will actually be positive depends on the $T$ and $C$ used to evaluate the rules. Therefore, only the fuzzy numbers $O_{K-m-j+1}$ in $O$, $0 \le j \le n + 1$, may have positive membership values $\Delta_{m+j}$. A "reasonable" defuzzifier $\delta_2$ will be a function of the $\Delta_{m+j}$ and $O_{K-m-j+1}$, for $0 \le j \le n + 1$, and its value will be a number in: (1) $[w(K - m - n - 1), w(K - m + 2)]$ if $2 \le m \le K - (n + 2)$; (2) $[w(K - n - 2), 1]$ if $m = 1$; and (3) $[-1, w(n + 3)]$ if $m = K - (n + 1)$. It seems to us that all reasonable defuzzifiers will produce a value in an interval $[a, b]$ corresponding to union of the supports of all the $O_l$ in $O$ the membership value of which can be positive. No specific value for $\delta_2$ need be identified for our results in the next section. This general defuzzifier $\delta_2$ contains $\delta_0$ and $\delta_1$ and also contains the center of gravity defuzzifier and the maximal membership defuzzifier.

The defuzzified output from the fuzzy controller at times $t = \Delta$, $2\Delta, \ldots$ will be $\delta = \delta_0$, $\delta = \delta_1$ or $\delta = \delta_2$. The defuzzified output is a function of the inputs $r^{(i)}$ and we wish to find an exact expression for this function. However, this is a very difficult task in general so instead we find the limiting form of this function as $N$ grows without bound.

## 2. *Main results*

Let

$$\mathscr{L} = -\sum_{i=0}^{n} r^{(i)}/(n + 1). \tag{10}$$

*Theorem* 1.

$$\lim_{N\to\infty} |\delta_0 - \mathscr{L}| = 0.$$

*Proof.* Given $r^{(i)} \in [-1, 1]$, $0 \le i \le n$, they determine the $j(i) \in \{1, 2, \ldots, 2N\}$ and let $c$ be the sum of the $r^{(i)}$ and

$$m = \sum_{i=0}^{n} j(i) - n.$$

The $\Delta_l$ can be positive only for $l = m, m + 1, \ldots, m + n + 1$ since, for each $i$, $\mu_j^{(i)}$ can be positive only for $j = j(i)$ and $j = j(i) - 1$. Let

$$P_j = \frac{\Delta_{m+j}}{\sum_{j=0}^{n+1} \Delta_{m+j}} \tag{11}$$

for $0 \le j \le n + 1$.

Simplifying equation (6) we find that $\delta_0 - \mathscr{L}$ is equal to

$$1 - \frac{m}{(n + 1)N} + \frac{1}{(n + 1)N} \sum_{j=0}^{n+1} (1 - j)P_j + \frac{c}{n + 1}. \tag{12}$$

The third term in equation (12) approaches zero as $N \to \infty$ because

$$-n \le \sum_{j=0}^{n+1} (1 - j)P_j \le 1. \tag{13}$$

We must now relate $c$ to $m$ in equation (12).

Let $\bar{r}^{(i)} = r^{(i)} - v(j(i))$, for $0 \le i \le n$. If $\bar{c}$ is the sum of the $\bar{r}^{(i)}$ we see that

$$c = \bar{c} + \sum_{i=0}^{n} v(j(i)). \tag{14}$$

Now substitute $[j(i) - (N + 1)]/N$ for $v(j(i))$ in equation (14) and then the term

$$1 - \frac{m}{(n + 1)N} + \frac{c}{n + 1} \tag{15}$$

from equation (12) becomes

$$\frac{\bar{c}}{n+1}-\frac{1}{(n+1)N}. \tag{16}$$

The expression in equation (16) approaches zero as $N\to\infty$ because

$$0\le\frac{\bar{c}}{n+1}\le\frac{1}{N}. \tag{17}$$

Hence, $\delta_0\to\mathscr{L}$ as $N\to\infty$.

*Theorem* 2.

$$\lim_{N\to\infty}|\delta_1-\mathscr{L}|=0.$$

*Proof.* Let $\alpha_l=w(l)+\epsilon_l$, where $|\epsilon_l|\le[(n+1)N]^{-1}$, for $1\le l\le K$. From the proof of Theorem 1 we see that

$$|\delta_1-\mathscr{L}|\le|\delta_0-\mathscr{L}|+\left|\sum_{j=0}^{n+1}P_j\epsilon_{K-m-j+1}\right|. \tag{18}$$

From Theorem 1 we know that $|\delta_0-\mathscr{L}|\to 0$ as $N\to\infty$. The other term on the right-hand side of equation (18) also approaches zero as $N\to\infty$ because

$$\left|\sum_{j=0}^{n}P_j\epsilon_{K-m-j+1}\right|\le\frac{1}{(n+1)N}. \tag{19}$$

Hence $\delta_1\to\mathscr{L}$ as $N\to\infty$.

*Theorem* 3.

$$\lim_{N\to\infty}|\delta_2-\mathscr{L}|=0.$$

*Proof.* There are three cases to consider as $N\to\infty$: (1) $2\le m\le K-(n+2)$; (2) $r^{(i)}=-1$ all $i$; and (3) $r^{(i)}=1$ all $i$. As $N$ grows $m$ will be changing and $m$ can remain equal to one only if $r^{(i)}=-1$ all $i$. Also, $m$ can remain equal to its maximum value of $K-(n+1)$ as $N$ grows only if $r^{(i)}=1$ all $i$.

*Case* 1. There are unique $\lambda_j\ge 0$, $-1\le j\le n+2$, at most two can be positive and their sum equals one, so that

$$\delta_2=\sum_{j=-1}^{n+2}\lambda_j w(K-m-j+1). \tag{20}$$

That is, $\delta_2$ is some convex combination of the central values of $O_{K-m-j+1}$ for $-1\le j\le n+2$. The $\lambda_j$ are functions of the $\Delta_{m+j}$ and $O_{K-m-j+1}$, for $-1\le j\le n+2$. If we alter the values of the $r^{(i)}$, or $N$, then possibly new values of the $\lambda_j$ in equation (20) must be calculated.

To show how the $\lambda_j$ are determined fix some values of the $r^{(i)}$, compute the $\Delta_{m+j}$ for $0\le j\le n+1$, and evaluate $\delta_2$ as some number in $[w(K-m-n-1), w(K-m+2)]$. Now $\delta_2$ will belong to some subinterval $[w(K-m-j), w(K-m-j+1))$ for $0\le j\le n+1$, or $\delta_2$ will belong to $[w(K-m+1), w(K-m+2)]$ for $j=-1$. Let $j=j^*$ denote the correct interval. Then we set $\lambda_j=0$ if $j\ne j^*$ or $j\ne j^*+1$ and determine $\lambda_{j^*}$ so that

$$\lambda_{j^*}w(K-m-j^*)+(1-\lambda_{j^*})w(K-m-j^*+1)=\delta_2. \tag{21}$$

From the proof of Theorem 1 we see that

$$\delta_2-\mathscr{L}=\left[1-\frac{m}{(n+1)N}+\frac{c}{n+1}\right]+\frac{1}{(n+1)N}\sum_{j=-1}^{n+2}(1-j)\lambda_j. \tag{22}$$

We know that the first term in equation (22) approaches zero as $N\to\infty$ from equation (16). The second term also approaches zero as $N\to\infty$ because

$$-(n+1)\le\sum_{j=-1}^{n+2}(1-j)\lambda_j\le 2. \tag{23}$$

*Case* 2. Now $m=1$ for all $N\ge 1$ and $\delta_2-\mathscr{L}$ simplifies to

$$\delta_2-\mathscr{L}=\left[1+\frac{c}{n+1}\right]-\frac{1}{(n+1)N}\sum_{j=0}^{n+2}j\lambda_j \tag{24}$$

using equation (20) to represent $\delta_2$ with $j$ starting at zero instead of minus one. The first term in equation (24) is zero because in this case $c=-1(n+1)$. The second term in equation (24) approaches zero as $N\to\infty$ because

$$0\le\sum_{j=0}^{n+2}j\lambda_j\le(n+2). \tag{25}$$

*Case* 3. Now $m=k-(n+1)$ for all $N\ge 1$ and $\delta_2-\mathscr{L}$ simplifies to

$$\delta_2-\mathscr{L}=\left[-1+\frac{c}{n+1}\right]+\frac{1}{(n+1)N}\sum_{j=-1}^{n+1}(n+1-j)\lambda_j \tag{26}$$

using equation (20) again to represent $\delta_2$ with $j$ ranging from $-1$ to $n+1$. The first term in equation (26) is zero since in this case $c=(n+1)$. The second term in equation (26) approaches zero as $N\to\infty$ because

$$0\le\sum_{j=-1}^{n+1}(n+1-j)\lambda_j\le n+2. \tag{27}$$

It is now obvious that "linear fuzzy control rules" is a sufficient condition for $\delta_i\to\mathscr{L}$ as $N\to\infty$, $i=0, 1, 2$, but it is not a necessary condition. We do not know of any necessary and sufficient condition on the structure of the fuzzy control rules so that $\delta_i\to\mathscr{L}$ as $N\to\infty$ for $i=0, 1, 2$.

## 3. *Example*

Let us consider two controller inputs called error $=e=r^{(0)}$ and rate $=r=r^{(1)}$. For three inputs the rule tables are three-dimensional and more difficult to present.

We start with three fuzzy numbers, called Negative, Zero, and Positive the central values of which are $-1$, 0, 1, respectively, for both $e$ and $r$. Table 1(a) gives the linear control rules. There will be five fuzzy numbers in the Output fuzzy set the central values of which are $-1$, $-0.5$, 0, 0.5, 1 and having linguistic names Negative-large, Negative, Zero, Positive, and Positive-large, respectively. The linearity of the rule table is evident in that the conclusions are the same down a diagonal parallel to the southwest to northeast main diagonal. Two of the five rules given in Table 1(a) are

$\mathscr{R}_2$: If ([error = Negative] AND [rate = Zero]) OR ([error = Zero] AND [rate = Negative]), then Output = $O_4$ = Positive (28)

and

$\mathscr{R}_3$: If ([error = Negative] AND [rate = Positive]) OR ([error = Zero] AND [rate = Zero]) OR ([error = Positive] AND [rate = Negative]), then Output = $O_3$ = Zero. (29)

Next let us increase to five fuzzy numbers for both $e$ and $r$. The central values of these fuzzy numbers are $-1$, $-0.5$, 0, 0.5, 1 having linguistic titles Negative-large, Negative, Zero, Positive, Positive-large, respectively. There are nine fuzzy numbers in Output with central values $-1$, $-0.75$, $-0.50$, $-0.25$, 0, 0.25, 0.50, 0.75, 1 associated with linguistic variables Negative-very large, Negative-large, Negative-medium, Negative-small, Zero, Positive-small, Positive-medium, Positive-large, Positive-very large, respectively.

TABLE 1(a). RULE TABLE FOR FIVE LINEAR FUZZY CONTROL RULES

| | | Rate | |
|---|---|---|---|
| Error | Negative | Zero | Positive |
| Negative | Positive-large | Positive | Zero |
| Zero | Positive | Zero | Negative |
| Positive | Zero | Negative | Negative-large |

TABLE 1(b). RULE TABLES FOR NINE LINEAR CONTROL RULES

| | Rate | | | | |
|---|---|---|---|---|---|
| Error | Negative-large | Negative | Zero | Positive | Positive-large |
| Negative-large | Positive-very large | Positive-large | Positive-medium | Positive-small | Zero |
| Negative | Positive-large | Positive-medium | Positive-small | Zero | Negative-small |
| Zero | Positive-medium | Positive-small | Zero | Negative-small | Negative-medium |
| Positive | Positive-small | Zero | Negative-small | Negative-medium | Negative-large |
| Positive-large | Zero | Negative-small | Negative-medium | Negative-large | Negative-very large |

Table 1(b) presents the linear control rules. The linearity is apparent from the same conclusions along any diagonal parallel to the southwest to northeast main diagonal. There are nine linear control rules in Table 1(b) and the one with conclusion Negative-small is

$\mathcal{R}_6$: If ([error = Negative] AND [rate = Positive-large]) OR $\cdots$ OR ([error = Positive-large] AND [rate = Negative]), then Output = $O_4$ = Negative-small. (30)

We can next increase to seven fuzzy numbers for $e$ and $r$ and then we will have 13 linear control rules presented in a $7 \times 7$ rule table. If we increase further to nine fuzzy numbers there are 17 linear control rules given in a $9 \times 9$ table. The theorems in the previous section imply that the defuzzified output $\delta$ approaches $-(e+r)/2$ as the rule table grows for any "reasonable" defuzzifier and for any fuzzy logic used to evaluate the rules.

### 4. *Conclusions*

Throughout this section we assume that the fuzzy controller has linear fuzzy control rules and some reasonable defuzzifier $\delta$ (such as $\delta_0$, $\delta_1$, $\delta_2$) all defined in the Introduction.

We have shown that, for any type of unimodal fuzzy number used to define the linguistic variables in the fuzzy control rules, for any type of fuzzy logic used to evaluate the fuzzy control rules, and for any reasonable defuzzifier, the defuzzified output from the fuzzy controller approaches a linear function of its input as the number of fuzzy control rules grow. Of course, for a small number of rules $\delta$ need not be a linear function of the inputs to the fuzzy controller. However, there are situations where $\delta$ can be a linear function of the inputs $r^{(i)}$ even for as few as three fuzzy numbers ($N = 1$) used for each input. We showed that (Buckley and Ying, 1989)

$$\delta_0 = -\sum_{i=0}^{n} r^{(i)}/(n+1) \tag{31}$$

for all $N \geq 1$ if triangular fuzzy numbers are used for each input, AND equals probabilistic AND, and OR is the Lukasiewicz OR. More research is needed to determine when $\delta$ will be a linear function of the inputs $r^{(i)}$ for small values of $N$.

Suppose we have two inputs ($n = 1$) and let $e$ = error = $r^{(0)}$ and $r$ = rate = $r^{(1)}$. Then

$$\delta \approx -\frac{e+r}{2} \tag{32}$$

for large $N$ and the defuzzified output is approximately equal to that of the PI controller. If we have three inputs ($n = 2$), let $d$ = the rate of change of rate = $r^{(2)}$, and then

$$\delta \approx -\frac{e+r+d}{3} \tag{33}$$

for large $N$. So for three inputs and sufficiently large $N$, the defuzzified output is approximately equal to that of the PID controller. One might believe that if we increase the number of rules we will be "fine tuning" the fuzzy controller and hence obtain better control of the process. This is certainly true if the process can be controlled by

$$u(t) = u(t-\Delta) - \frac{\sum_{i=0}^{n} r^{(i)}}{n+1}(\Delta). \tag{34}$$

However, if the process is not controlled by the linear controller given in equation (34), then increasing the number of rules in the fuzzy controller will also fail to control the process. In fact, the fuzzy controller could do possibly better for small $N$ than for large $N$ if equation (34) is not suitable for controlling the process. Further research is needed to determine the nonlinearities of $\delta$ for small $N$ and how this might be exploited in fuzzy controllers.

Our main results in Section 2 were limiting results as $N \to \infty$ so we would also like to know something about the rate of convergence. For example, find the smallest $N$, say $N^*$, so that

$$|\delta - \mathcal{L}| < 0.001 \tag{35}$$

for $N \geq N^*$. Of course, $N^*$ may depend on the number of inputs, the types of fuzzy numbers, the defuzzifier, and the $t$-norms and co-$t$-norms used to evaluate the rules. We do have the following result:

$$|\delta_0 - \mathcal{L}| \leq 0.0505/N \tag{36}$$

when there are two inputs, the fuzzy numbers are triangular fuzzy numbers, AND = min and OR = max. More research is needed to obtain further results in this area.

We mentioned in the Introduction that

$$\delta \doteq F(r^{(0)}, \ldots, r^{(n)}) \tag{37}$$

or that $\delta$ is some (unknown) function $F$ of the inputs. We would like to find, for small values of $N$, a specific formula for $F$. We have some preliminary results for two inputs, $\delta_0$, the fuzzy numbers are either triangular or trapezoidal, AND = min and OR = max. More research is also needed to obtain further results in this area.

At the other extreme of $N \to \infty$ we have the special case of only two fuzzy numbers ("negative" and "positive") used for each input (Ying *et al.*, 1988). In this case we have only three fuzzy control rules and this special case also appears worthy of further investigation.

### *References*

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